Boom-Bust Cycles of Learning, Investment and Disagreement

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Abstract

An asymmetric cycle of investment and beliefs emerges when payoffs reveal information about the state of the world and an agent (Follower) only learns about another agent's (Loner's) payoffs from his actions. Entry to the market is gradual as Follower "free-rides" on Loner's information and waits to see him stay in the market before entering himself. However, exit tends to be abrupt: once Loner exits the market, Follower does the same as exit is a strong indicator of adverse conditions. After leaving the market, agents do not observe payoffs and cannot tell if conditions have improved so activity is resumed long after it crashes.

The asymmetry in the cycle is magnified when information is public. If Follower observes Loner's payoffs and not just his actions, he is more likely to defer his entry compared to the benchmark model. Finally, model simulations show a positive correlation between investment and dispersion of beliefs which is largely attributed to the learning mechanism in the model.

1 Introduction

Rational agents rely on available information to choose optimal actions, yet often agents' actions also generate information. Specifically, this is true in situations where active investment provides excess information regarding potential profitability. For example, a venture capital fund will learn more about the technology under development and about the business plans of a startup once the fund invests in it; a firm will learn more about potential revenue at a new geographic location after it opens a new branch in the area. While investment is a source for potential monetary gains in these settings, it is also a tool for agents to acquire information regarding the asset in which they are investing. The additional role of investment increases agents' incentives to enter the market. However, if agents are able to observe decisions by other rational agents, the incentive to learn by investment is moderated as agents "free-ride" on the information acquired by others.

This paper examines the implications of the two way connection between information and real actions on the dynamics and comovements of investment, learning and beliefs. To highlight the mechanisms in play I focus on a setting of two agents, Loner and Follower, where both can invest in a similar asset. When choosing to invest, an agent receives a privately observed payoff that, conditional on the state of the world, is drawn independently from the other agent's payoff. The two agents differ in the ability to observe each other's actions. Loner is assumed to be "isolated" - he cannot observe the actions of Follower. However, Follower can observe Loner's investment decisions through which he can learn about Loner's private signals.

Equilibrium in the model exhibits two main features. First, an asymmetric dynamic of beliefs and investment. While agents' entry to the market is gradual, the exit is usually abrupt.

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Second, recovery is slow, meaning that after activity crashes it takes a long time for agents to resume investment. The basic mechanism that generates these results is as follows. Agents enter the market when they become optimistic enough regarding the state of the world (once their belief crosses a certain threshold). Entry is gradual, as Follower observes Loner's actions and thus learns about the state without engaging in active investment himself. This means that Follower prefers to "free-ride" on Loner's information and defer his entry. Once agents enter the market they start receiving payoffs which provide information about the state of the world and the profitability of investment. If an agent receives bad news he opts to exit the market. In a situation where Loner receives a negative signal before Follower and is seen leaving the market, Follower experiences a "Wile E. Coyote moment".[1] Like the cartoon character who runs off the cliff in pursuit of the Road Runner, only to find himself hovering in mid-air before crashing to the ground, Follower realizes that his optimism is groundless. He immediately exits the market, causing an abrupt crash in activity. The crash in activity cuts off the flow of information, slowing agents' reactions to improvements in the state of the world and making the "boom" in activity more gradual than the crash.

The paper is closely related to the literature on social learning in which agents' actions generate information (see Hörner and Skrzypacz (2016) for a recent survey). One class of models in this literature is that of "strategic bandit models" where the signals generated by agents' actions are publicly observed [Bolton and Harris (1999); Keller et al. (2005); Keller and Rady (2015)]. In another class of models signals are private, however, the action that generates them is irreversible [Chamley and Gale (1994); Caplin and Leahy (1994); Rosenberg et al. (2007); Moscarini and Squintani (2010); Murto and Välimäki (2011, 2013)]. Frick and Ishii (2015) construct a model with publicly observed signals and irreversible actions. My model differs from this strand of literature in two aspects. First, these models assume that the state of the world is constant while I allow for the state to change over time in order to study the dynamics of the main variables. Second, my model is the first, to the best of my knowledge, to tackle a setting of private information with reversible actions (agents are free to enter and exit the market in any period). The focus on reversible actions makes it possible to explore the dynamics of investment and learning without imposing any a-priori asymmetries in the model: any asymmetric dynamics that arise in my model are due to the information structure. The assumption of private signals turns out to have implications on these asymmetries and it also allows the studying of dispersion in posterior beliefs.

Adding these complexities to the model requires simplifying elsewhere in order to keep the model tractable. For this reason I take on the simplifying assumption of asymmetric roles between the two agents (Follower observes Loner but not vice versa) while the literature usually explores symmetric settings and equilibriums. The model should thus be viewed as a step toward a complete solution of a problem that so far has not been addressed in the literature. Nonetheless, it may also be relevant for settings in which agents' abilities to observe others are exogenous to their investment decisions. For example, when two firms that engage in the same market differ in size it is plausible that the small firm will be able to observe the big firm's actions but not vice versa. All the more so if one of the firms is private and the other is public and thus the latter is obligated to report its actions to shareholders. It may even be the case that two public firms are subjected to different disclosure requirements. In Israel, for example, new firms on the stock exchange are granted leniencies in mandatory disclosure to shareholders.

The paper also relates to the literature that models the feedback between information and macro or financial activity. The common feature of these models is that optimism boosts economic activity which in turn generates more public information. Thus, there is a larger flow of information and faster reaction to shocks in good times, when shocks tend to be adverse, than in bad times, when shocks tend to be positive. This mechanism creates asymmetric

[1] The phrase was coined by Krugman (2007) in reference to foreign investors financing what he viewed as an unsustainable U.S. current account deficit in the mid 2000s.

cycles - the market crashes quickly following a negative shock but recovery is gradual. In Veldkamp (2005) this mechanism is modeled in a credit market where investment generates information regarding the profitability of the project. Fajgelbaum et al. (2017) highlight the role of uncertainty in investors' decision-making and show that the interaction between information and investment leads to uncertainty traps with low activity and high uncertainty. My model features similar mechanisms, as information is more abundant in good times when activity is high. However, since my focus is on information channels I relax the simplifying assumptions made in these models such as public information, myopic agents or irreversible entry to market. An asymmetric informational cycle also arises in Zeira (1994, 1999) and in Van Nieuwerburgh and Veldkamp (2006). In these models the structure of the production function is such that changes in unobserved fundamentals are identified more accurately when production is high.

The second part of the paper focuses on the role of private information in generating the main results. Section 5 shows that regardless of the quality of information observed by Follower (i.e., Loner's signals or actions), the "free-riding" effect makes him enter the market after Loner. The results regarding abrupt exits and slow recoveries also hold, so asymmetry is maintained in the public information setting as well. However, the quality of information does affect the incentive and timing of Follower's entry. Directly observing Loner's signal makes it less costly for Follower to enter the market, compared to the baseline setting in which he observes Loner's actions. Follower's threshold for entry is thus lower in the public information setting. Nonetheless, in equilibrium he is more likely to enter the market *later*. The intuition is that Follower can now tell if Loner is staying in the market because he received good news or because he has not yet received bad news. In the private information setting these two alternatives are indistinguishable so Follower quickly becomes very optimistic when he observes Loner stay in the market, and so his belief crosses his threshold earlier. This means that public information generally defers entry to the market and amplifies the asymmetry of the cycle.

Finally, Section 6 demonstrates that disagreement, i.e., dispersion of beliefs, is pro cyclical: it is positively correlated with total investment. This is because when both agents are in the market, information is based more on private signals, making beliefs more dispersed. Once activity crashes, beliefs become perfectly aligned and remain that way until investment is resumed. The learning mechanism in the model plays an important role in generating this positive correlation and absent of it disagreement is less correlated with activity and may even be counter cyclical.

The rest of the paper is organized as follows. Section 2 describes the general setting of the model; Section 3 discusses theoretical results and simulations for a setting with a constant state of the world; Section 4 generalizes these results to a stochastic state setting; Section 5 analyzes the role of private information by comparing the benchmark model to a public information setting; Section 6 discusses the dynamics of disagreement in equilibrium, and Section 7 concludes.

2 General Setting

There are two possible states of the world $\theta \in \{G, B\}$ and time is discrete $t = 0, 1, 2...$. Initially, I assume that the state is constant over time and later I relax this assumption by allowing θ_t to follow a Markov Chain process. The structure of the model is such that symmetry is maintained wherever possible so that any asymmetric dynamics that arise are due solely to the information structure.

There are two risk neutral agents in the economy. Let μ_t^i denote the probability that agent i assigns to the state of the world being G at the beginning of period t. I assume that priors, $\{\mu_0^i\}$, are common knowledge. Both agents discount future earnings by the same factor $\delta \in (0, 1)$.

There is one asset in the economy and, conditional on the state θ, its returns are drawn independently across time and across agents. Specifically, in each period the investment pays 0 with probability p, and π^θ with probability $1-p$. This means that if an agent observes π^θ he learns the state of the world in that period, and a payoff of 0 teaches him nothing about the state (note that $1-p$, the probability of payment π^θ, is independent of θ). This distribution of payoffs is chosen to maintain the a priori symmetry in the model as it imposes no trend in beliefs.[2] If an agent decides to invest in the asset in period t, he incurs a cost c in that period (there is no fixed cost for investment) and his payoff is private information. If he chooses not to invest his payoff in that period is 0.[3]

Denote by x the expected payoff from active investment in a single period given state G, meaning $x = (1-p)\pi^G - c$. I assume that $x > 0$. In order to maintain symmetry between the two states of the world, I further assume that the expected payoff given state B is $-x$, meaning $(1-p)\pi^B - c = -x < 0$. This specifically means that the investment is profitable in state G but not in state B.

3 Constant State of the World

Assume that the state of the world θ is constant over time but initially unknown to agents.

3.1 Loner

As aforesaid, Loner is an isolated agent and cannot observe the other agent's actions. Note that the only relevant information for his decision making in period t is his belief μ_t. This belief is updated according to Bayes' rule after each period's payoff π_t is revealed (note that $\pi_t = 0$ if the agent did not invest or if his investment yielded a payoff of 0):

$$\mu_{t+1}(\pi_t) = \begin{cases} \mu_t & \text{if } \pi_t = 0 \\ 1 & \text{if } \pi_t = \pi^G \\ 0 & \text{if } \pi_t = \pi^B \end{cases}$$

The problem facing the agent is thus a dynamic optimal choice problem. Given a belief μ, let $V^N(\mu)$ and $V^I(\mu)$ denote the value gained by the agent if he chooses not to invest in the current period and to invest, respectively. The Bellman equation for the dynamic optimal choice problem is therefore:

$$V(\mu) = \max\{V^N(\mu), V^I(\mu)\}$$

where,

$$V^N(\mu_t) = \delta V(\mu_t)$$

$$V^I(\mu_t) = E_{\mu_t}\left[\pi_t + \delta V(\mu_{t+1})\right] - c =$$

$$\Pr(\pi_t = 0|\mu_t)\left[0 + \delta V(\mu_t)\right] + \Pr(\pi_t = \pi^G|\mu_t)\left[\pi^G + \delta V(1)\right] + \Pr(\pi_t = \pi^B|\mu_t)\left[\pi^B + \delta V(0)\right] - c =$$

$$p\delta V(\mu_t) + (1-p)\mu_t\left[\pi^G + \delta V(1)\right] + (1-p)(1-\mu_t)\pi^B - c = (2\mu_t - 1)x + \delta\left[pV(\mu_t) + (1-p)\mu_t\frac{x}{1-\delta}\right]$$

[2] "One-armed bandit" models for example usually feature a Poisson distribution with a rate that depends on the state of the world. However, the difference between rates imposes a drift in beliefs over time which governs affects behavior.

[3] This setting can be viewed as follows. There is an outside option that guarantees a fixed payoff r (which we normalize to 0). For the most part the investment is similar to the outside option and also pays r. The true nature of the investment is revealed only on rare occasions, with probability $1-p$, but in these occasions the payoff is either extremely high or extremely low, depending on the state of the world. These rare occasions are what determine if the investment is preferable to the outside option.

Note that if Loner chooses not to invest, he will not receive new information, and thus if it is optimal for him not to invest now he will never invest again. The Bellman equation can thus be written as:

$$V(\mu) = \max\left\{0\,,\,(2\mu-1)x + \delta\left[pV(\mu) + (1-p)\mu\frac{x}{1-\delta}\right]\right\} \tag{1}$$

I will now characterize V and show that Loner's optimal choice is to invest as long as his belief exceeds a certain threshold.

Definition 1. A function $f : [0,1] \to \mathbb{R}$ is **threshold-linear convex (TLC)** with threshold M if f is continuous, increasing and convex, and if there exists a threshold M such that f is linear on $[M, 1]$.

Proposition 1.

1. *There exists a unique value function $V : [0,1] \to \mathbb{R}$ that satisfies (1).*
2. *There exists a threshold belief $M^L = \frac{1-\delta}{2-\delta(1+p)}$ such that Loner invests if and only if $\mu \geq M^L$. Furthermore, V is TLC with threshold M^L (in fact, it is quasilinear).*

Proof. See Proposition 3 for the general case.

Note that the agent's threshold M^L is smaller than 0.5 which is the threshold of the myopic agent (an agent with a discount factor $\delta = 0$). This is due to the value of information embedded in active investment. At the threshold belief, the agent is willing to incur a cost of $(1-2M^L)x = \frac{\delta(1-p)x}{2-\delta(1+p)} > 0$ for the opportunity to learn about the state of the world.

Another important insight is that there are only three types of histories that will materialize in equilibrium:

1. **Continued investment**: If Loner's prior μ_0^L was higher than M^L then as long as he has not seen a payoff of π^B his belief is either μ_0^L or 1 and he continues to invest.
2. **Investment terminated**: If $\mu_0^L \geq M^L$ but at some point the agent observed π^B then he updated his belief to 0 and stopped investing.
3. **Never invested**: Of course, if $\mu_0^L < M^L$ the agent will never invest.

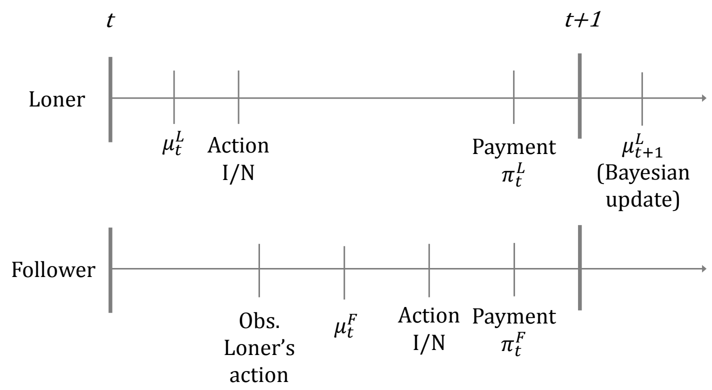
3.2 Follower

To understand the role of learning by other's actions, I now introduce a second agent to the model - Follower. Like Loner, Follower can invest in the asset described above. Conditional on the state this asset pays agents independent payoffs. While I assumed Loner can learn only by active investment, Follower also observes Loner's actions. These actions provide information regarding Loner's private signals and the state of the world. Specifically, in each period Follower observes Loner's investment decision before deciding on his own action. Figure 1 illustrates the timeline of investment decisions and payments for both agents.

As pointed out in Section 3.1, there are three types of histories of Loner's actions that Follower may observe in equilibrium. If Loner never invested then Follower is an "isolated" agent and his dynamic optimal choice problem is as described in Section 3.1. If Loner invested in the past but at some point stopped investing, then Follower concludes that Loner observed π^B. He immediately updates his own belief to 0 and stops investing (without ever resuming investment).

The interesting case is a history in which Loner invested in all previous periods. When observing Loner investing one more period, Follower becomes more optimistic and his belief is

Figure 1: Timeline of Investment Decisions and Payments



updated according to $\mu \mapsto \frac{\mu}{p+(1-p)\mu}$. Otherwise, his belief is updated to 0. If Follower does not invest himself then this is his only source of information. If he does invest, then with probability p he receives a payment of zero and is left to learn only by Loner's actions (as if he did not invest in the first place). With probability $1-p$ Follower's own signal reveals the state of the world and the information from Loner's action is redundant.

Formally, denote Follower's value function given Loner invested so far and given his own action a by U^a, $a \in \{I, N\}$. Then Follower's Bellman equation is given by:

$$U(\mu) = \max\{U^N(\mu), U^I(\mu)\} \quad \text{where} \tag{2}$$
$$U^N(\mu) = [p + (1-p)\mu]\delta U\left(\frac{\mu}{p+(1-p)\mu}\right)$$
$$U^I(\mu) = (2\mu - 1)x + pU^N(\mu) + (1-p)\mu\delta\frac{x}{1-\delta}$$

Proposition 2.

1. *There exists a unique value function $U : [0,1] \to \mathbb{R}$ that satisfies (2). This function is non-decreasing, continuous and convex.*

2. *$U(\mu) \geq V(\mu)$ for all $\mu \in [0,1]$, meaning that Follower has positive gains from observing Loner's actions.*

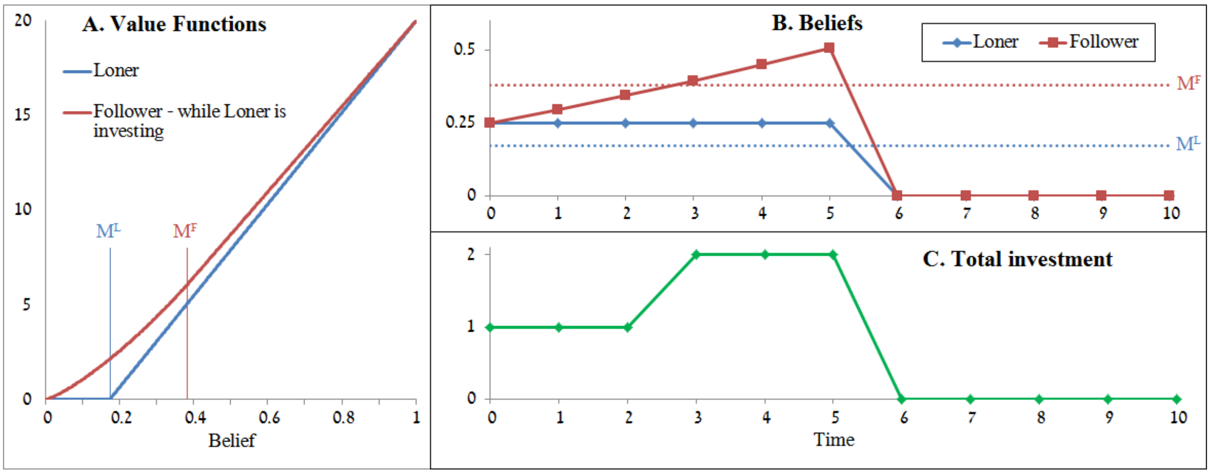
3. *There exists a threshold belief $M^F = \frac{1-\delta p}{2-\delta p(1+p)} \in (M^L, 0.5)$ such that Follower invests if and only if $\mu \geq M^F$, and $U(\cdot)$ is TLC with threshold M^F. Specifically,*

$$U(\mu) = \max\left\{[p+(1-p)\mu]\delta U\left(\frac{\mu}{p+(1-p)\mu}\right), \quad \left[\frac{x}{1-\delta} + \frac{x}{1-\delta p^2}\right]\mu - \frac{x}{1-\delta p^2}\right\} \tag{3}$$

Proof. See Proposition 4 for the general case.

The fact that $M^F > M^L$ is due to Follower's "free-riding". At the belief M^L an isolated agent would have started investing but since Follower has the privilege of observing Loner, the gain from not investing is higher which makes this option more attractive. However, M^F is still lower than the threshold of the myopic agent, 0.5, since investing always provides excess information regarding the state of the world.

Figure 2: The Constant-State Model - Numeric Approximation of Value Functions and Simulation of Beliefs and Total Investment when Loner Receives Bad News



Notes: Panel A depicts numeric approximations of Loner's and Follower's value functions for the parameters $x = 1$, $\delta = 0.95$ and $p = 0.8$. Panels B and C depict the dynamics in a simulation of this model in which both agents have a prior belief of $0.25 \in (M^L, M^F)$ and no news arrive until period 4 (both agents receive payments of zero). In period 5 Loner observes π^B and leaves the market in the following period. His exit is immediately followed by Follower's departure from the market.

3.3 Dynamics

The simple setting of constant state provides insights that will carry through to the stochastic settings. As I will demonstrate, Follower's "free-riding" induces asymmetric dynamics in beliefs and investment. While the ascent of beliefs and investment occurs gradually, their decline can be abrupt. Furthermore, agents' optimism is accompanied by dispersion in beliefs, while pessimistic beliefs are aligned.

An example of the dynamics arising in this model is depicted in Figure 2 (general results are established in Section 4.2). In this example both agents have a prior belief that is above Loner's threshold but below Follower's. Thus, Loner starts investing in period 0. In the first five periods Loner does not receive any information, meaning he does not observe π^θ, his belief remains unchanged and he continues to invest. As for Follower, initially his belief is not high enough to evoke investment and he prefers to passively observe Loner ("free-riding" effect). Observing Loner staying in the market over time makes Follower more and more optimistic until his belief crosses his own threshold and he enters the market. In Follower's point of view, "no news is good news", meaning that observing Loner staying in the market makes him more confident that the state of the world is G.

In the example of Figure 2 both agents receive payments of zero until period 4, meaning that no new information arrives to the market. Nonetheless, Follower becomes highly optimistic and his and Loner's beliefs grow further and further apart. In other words, the disagreement between the agents intensifies. In period 5 Loner observes π^B which results in him updating his belief to 0 in period 6 and exiting the market. At this point Follower experiences a "Wile E. Coyote moment". Like the cartoon character who runs off the cliff in pursuit of the Road Runner only to find himself hovering in mid-air, Follower suddenly realizes that his optimistic belief is groundless and he immediately exits the market. From this point on, both agents share the same pessimistic belief, i.e., disagreement vanishes, and investment is halted indefinitely.

4 Stochastic State

In this section I explore a more general setting in which the state of the world varies over time. Specifically, I assume that θ_t is a stationary Markov Chain process with the state space $\{G, B\}$ and conditional probabilities $\Pr(\theta_t = \theta | \theta_{t-1} = \theta) = \lambda$ for $\theta \in \{G, B\}$ and for some $\lambda \in (0.5, 1]$. This means that the state of the world is persistent (in each period there is a higher probability to remain in the same state than to switch) but transition to the other state is equally likely under both states. With no other information, beliefs evolve according to $\mu \mapsto \lambda\mu + (1-\lambda)(1-\mu) = (2\lambda - 1)\mu + 1 - \lambda$ and any initial belief converges over time to the steady state 0.5.

4.1 Loner and Follower's Value Functions

The generalization of Loner's value function to the stochastic state setting is straightforward. It only requires adjusting the evolution of beliefs to the possibility of state transition. Specifically, Loner's value function takes the following form:

$$\begin{aligned} V(\mu) &= \max\{V^N(\mu), V^I(\mu)\} \quad \text{where,} \\ V^N(\mu) &= \delta V\Big((2\lambda - 1)\mu + 1 - \lambda\Big) \\ V^I(\mu) &= (2\mu - 1)x + pV^N(\mu) + \delta(1-p)[\mu V(\lambda) + (1-\mu)V(1-\lambda)] \end{aligned} \tag{4}$$

The following proposition characterizes the function V and shows that there exists a unique threshold belief $M^L \in (1-\lambda, 0.5)$ such that Loner invests only when his belief exceeds this threshold.

Proposition 3. ***(Properties of Loner's Value Function)***

1. *There exist unique value functions $V : [0,1] \to \mathbb{R}_+$ that satisfies (4). This function is non-decreasing, continuous and convex.*
2. *There exists a threshold belief $M^L \in (1-\lambda, 0.5)$ such that Loner invests if and only if $\mu \geq M^L$, and his value function is TLC with threshold M^L. Specifically,*

$$V(\mu) = \max\left\{\delta V\Big((2\lambda - 1)\mu + 1 - \lambda\Big), \quad \alpha^L\mu + \beta^L\right\}$$

where $\alpha^L = \frac{x(2-\delta-\delta p)-(1-\delta)(1-p)V(0)}{(1-\delta\lambda)(1-\delta p)+\delta p(1-\delta)(1-\lambda)}$, $\beta^L = \frac{x-(1-\delta\lambda)\alpha^L}{1-\delta}$, *and* $M^L = \frac{x-(1-p)[V(0)-\beta^L]}{(1+p)x-(1-p)[V(0)-\beta^L]}$

Proof. All proofs are deferred to the Appendix.

Note that Loner's belief evolves depending on periodic payoffs as follows:

$$\mu^L_{t+1}(\pi_t) = \begin{cases} (2\lambda - 1)\mu^L_t + 1 - \lambda & \text{if } \pi_t = 0 \\ \lambda & \text{if } \pi_t = \pi^G \\ 1 - \lambda & \text{if } \pi_t = \pi^B \end{cases}$$

Further note that if $\mu^L_t \geq M^L$ then $(2\lambda - 1)\mu^L_t + 1 - \lambda \geq M^L$ as well.[4] Therefore, Loner will continue to invest as long he has not received bad news, i.e., has not observed π^B. Once he observes π^B his belief is updated in the following period to $1 - \lambda$ and he stops investing. While $\mu^L_t < M^L$ Loner does not invest and his belief evolves according to $\mu_{t+1} = (2\lambda - 1)\mu_t + 1 - \lambda$. This means that n periods after Loner observed π^B his belief is $0.5 - 0.5(2\lambda - 1)^n$. Since this

[4]If $\mu \geq 0.5$ then $(2\lambda - 1)\mu + 1 - \lambda \geq 0.5$, and if $M^L \leq \mu < 0.5$ then $(2\lambda - 1)\mu + 1 - \lambda > \mu$.

process converges to 0.5, eventually Loner's belief crosses his threshold and he starts investing. More precisely, after observing π^B Loner will not invest for $\bar{n} \equiv \lfloor \frac{\ln(1-2M^L)}{\ln(2\lambda-1)} \rfloor$ periods and will resume investment in the $(\bar{n}+1)^{\text{th}}$ period.

This means that in equilibrium, in each period t there are two types of states that are relevant to Follower:

1. **Loner will resume investment in n periods ($n \in \{1, ..., \bar{n}\}$):** This means that Loner did not invest for $\bar{n}+1-n$ periods up until and including period t. In this case Loner's action in $t+1$ is known for certain (if $n=1$ he will invest in the following period, and if $n>1$ he will not) so it will be uninformative and Follower will update his belief according to $\mu^F_{t+1} = (2\lambda-1)\mu^F_t + 1 - \lambda$.

2. **Loner invested in t:** In this case Follower's belief μ^F_{t+1} is distributed according to the probability that Loner will invest another period or will stop investing (see proof of Proposition 4 for a formal definition).

Formally, the possible states that Follower may be facing can be categorized by a couple (μ^F_t, n_t) where μ^F_t is Follower's belief, and $n_t \in \{0, 1, ..., \bar{n}\}$ is the number of periods remaining until Loner resumes investment ($n_t = 0$ indicates that Loner invested in t).

Follower updates his belief μ^F_t according to his private signals and Loner's actions as in the constant state setting, with the additional adjustment to the possibility of state transition. The prominent difference from the constant state setting is in the evolution of n_t. While Loner is investing ($n_t = 0$), Follower's expected utility depends on whether Loner will remain in the market for another period. If he does, then the state variable n_{t+1} remains zero. Otherwise, Follower's deduces that Loner observed π^B in period t and n_{t+1} updates to $\bar{n}$. If $n_t \geq 1$ then necessarily $n_{t+1} = n_t - 1$ (the "counter" for the number of periods until Loner resumes investment decreases by 1). Thus, Follower's value function has a recursive structure:

$$U(\mu, n) = \max\left\{U^N(\mu, n), U^I(\mu, n)\right\} \text{ where,} \tag{5}$$

$$\begin{aligned}
U^N(\mu, 0) =& [p + (1-p)\mu]\delta U\left((2\lambda - 1)\frac{\mu}{p + (1-p)\mu} + 1 - \lambda, 0\right) + (1-p)(1-\mu)\delta U(1-\lambda, \bar{n}),\\
U^I(\mu, 0) =& (2\mu - 1)x + pU^N(\mu, 0) +\\
& (1-p)\mu\delta U(\lambda, 0) + (1-p)(1-\mu)\delta\Big[pU(1-\lambda, 0) + (1-p)U(1-\lambda, \bar{n})\Big]
\end{aligned}$$

and for $n \in \{1, ..., \bar{n}\}$,

$$\begin{aligned}
U^N(\mu, n) =& \delta U\Big((2\lambda-1)\mu + 1 - \lambda, n-1\Big),\\
U^I(\mu, n) =& (2\mu - 1)x + pU^N(\mu, n) + (1-p)\delta\Big[\mu U\big(\lambda, n-1\big) + (1-\mu)U\big(1-\lambda, n-1\big)\Big]
\end{aligned}$$

An important difference between the stochastic state setting and the constant state setting is that in the latter if an agent stopped investing he will never resume investment. However, when the state is stochastic even a belief of zero will eventually converge to 0.5 in which it is clearly optimal for the agent to invest. Inter alia, this implies that $V(0)$ and $U(0, n)$ are no longer equal to zero.

The following proposition characterizes Follower's value function and shows that for every n there exists a unique threshold belief $M^F_n \in (1-\lambda, 0.5)$ such that Follower invests only when his belief exceeds this threshold.

Proposition 4. ***(Properties of Follower's Value Function)***

1. *There exists a unique value function $U : [0,1] \times \{0,...,\bar{n}\} \to \mathbb{R}_+$ that satisfies (5). This function is non-decreasing, continuous and convex in the first argument.*

2. *$U(\mu, n) \geq V(\mu)$ for all $n = 0, ..., \bar{n}$ and $\mu \in [0,1]$.*

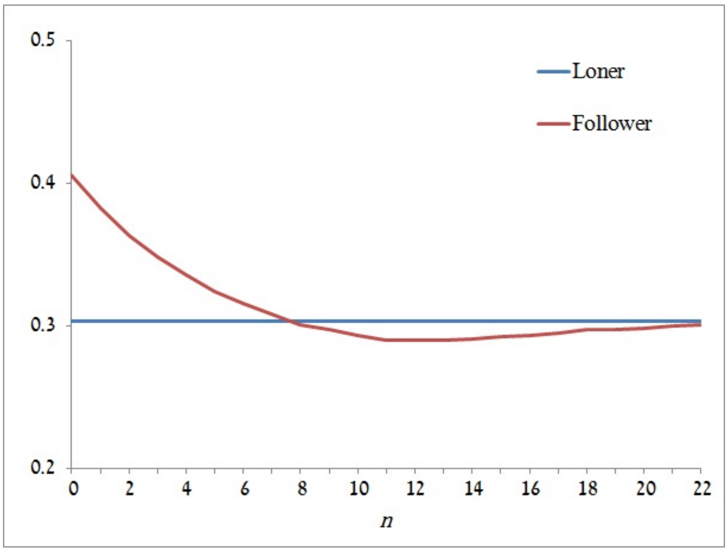
3. *For all $n = 0, ..., \bar{n}$ there exists a unique threshold belief $M_n^F \in (1-\lambda, 0.5)$ such that Follower invests in state (μ, n) if and only if $\mu \geq M_n^F$, and $U(\mu, n)$ is TLC .*

4. *Define $U(\mu, n)$ as in (2) for all $n \in \mathbb{N}$ (i.e, extend the definition for $\bar{n}+1, \bar{n}+2, ...$) then $\lim_{n\to\infty} U(\mu, n) = V(\mu)$. This means that as Loner's entry is further in the future, Follower acts as an isolated agent.*

Figure 3 depicts numeric approximations of Loner and Follower's thresholds as a function of n - the number of periods until Loner is known to enter the market in equilibrium. The figure illustrates that when Loner's entry is close (n is small) Follower's incentive to invest is lower than Loner's and his threshold exceeds M^L. This is because free information will become available in the near future. As n increases this promise becomes less valuable and Follower's incentive to actively invest increases (his threshold decreases). Note that for some values of n Follower's threshold is lower than M^L. This is because the promise of Loner's entry makes it less costly for Follower to enter the market and receive a bad outcome. To be more concrete, consider a belief of 0.3 when $n = 12$. Note that $0.3 \in (M_{12}^F, M^L)$. An isolated agent like Loner that is facing a belief of 0.3 prefers to stay out of the market since his entry will likely result in a low signal that will keep him out of the market for a long time (22 periods). However, Follower knows that in any case Loner will enter the market in 12 periods. When this occurs, Follower will receive information that will potentially be positive. Thus, Follower's expected loss from entering today and receiving bad news is lower. When n is large the promise of future information has a very small effect on Follower's incentives and his threshold for entry approaches that of an isolated agent M^L.

Even though Follower's threshold may be lower than Loner's for some values of n, Section 4.2 will show that for λ close enough to 1 Follower will enter the market after Loner.

Figure 3: Numeric Approximation of Agents' Investment Thresholds



Notes: The figure depicts numeric approximations of Loner and Follower's thresholds (M^L and M_n^F, respectively) for the parameters $x = 1$, $\delta = 0.95$, $p = 0.8$ and $\lambda = 0.98$. Thresholds are depicted as a function of n, the number of periods until Loner is known to return to the market.

4.2 Asymmetric Dynamics

In this section I discuss the dynamics that arise in equilibrium. The pattern of investment in equilibrium tends to be asymmetric. When the state of the world is persistent enough, entry to the market is gradual, meaning that Loner always enters before Follower. However, exit from the market is generally abrupt, meaning that both agents leave the market at the same time. Furthermore, after both agents leave recovery takes a long time, meaning that investment is re-initiated long after it ceased.

First, I will establish that entry to the market is gradual when λ is close enough to 1.[5] This is due to Follower's "free-riding", meaning that in equilibrium he will prefer to wait and see Loner stay in the market for a while before entering himself:

Proposition 5. ***(Free-Riding and Gradual Entry)***
Assume both agents share the same belief and Loner is out of the market. If λ is close enough to 1 then Follower will enter the market strictly after Loner.

Next, I show that exit from the market is more likely to be abrupt than gradual, meaning that the probability that both agents exit in the same period in equilibrium is higher than the probability that they exist one after the other. Note that the former event occurs whenever Loner observes a bad signal and exits the market, since Follower will immediately update his belief to $1-\lambda$ and exit as well. The latter event occurs only if Follower observes bad news but Loner observes an uninformative signal.

Proposition 6. ***(Abrupt vs. Gradual Exit)***
Assume both agents were in the market in period $t-1$ and some agent leaves the market in period t. The probability of an abrupt exit in t (both agents leave the market) is $\frac{1}{1+p}$ while the probability that only one agent exits is smaller and equals $\frac{p}{1+p}$.

Proof. In equilibrium, an exit occurs in t with probability $(1-p^2)\mathbb{1}_{\{\theta_t=B\}}$ where $\mathbb{1}_{\{\theta_t=B\}}$ is the indicator function that equals one if $\theta_t = B$ and zero otherwise. Given an exit at t, the probability that Loner exits and thus the exit is abrupt, equals $\frac{(1-p)\mathbb{1}_{\{\theta_t=B\}}}{(1-p^2)\mathbb{1}_{\{\theta_t=B\}}} = \frac{1}{1+p}$. □

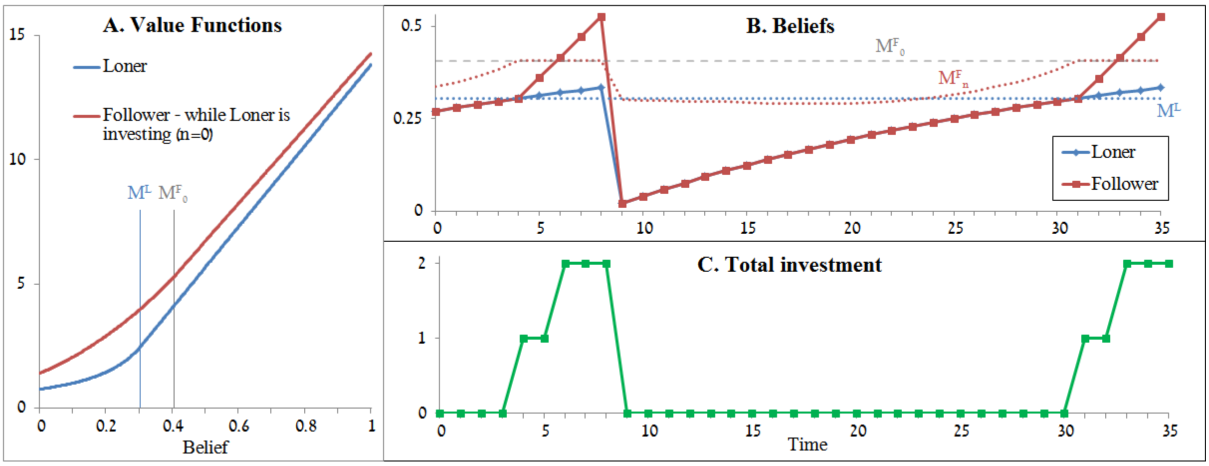
Finally, after both agents leave the market recovery will be slow. Specifically, it will take $\bar{n}$ periods for Loner to re-enter the market and if λ is high, Follower will enter even longer after that.

Corollary 7. ***(Slow Recovery)***
If λ is close enough to 1, no agent will enter the market in the $\bar{n} = \lfloor \frac{\ln(1-2M^L)}{\ln(2\lambda-1)} \rfloor$ periods after Loner exited.

An example of the asymmetric dynamics in the model is depicted in Figure 4. In this example, both agents have a prior belief that is lower than Loner's threshold so initially they are both out of the market. As time progresses their belief updates according to the Markovian process until it crosses M^L in period 4. At this point, Loner enters the market but Follower prefers to "wait and see" how events unfold. In this example, Loner does not receive any news in periods 4-7 so he remains in the market. Observing Loner continuing to invest makes Follower more and more optimistic until his belief crosses his threshold and he enters the market in period 6. Note that this dynamic of gradual entry occurs without any actual news being received in the market. In period 8, Loner is assumed to observe an adverse signal, causing him to update

[5]The focus on large λ's is for two reasons. First, unlike the case of $\lambda = 1$, for $\lambda < 1$ thresholds cannot be derived analytically and the approximation to $\lambda = 1$ allows to characterize them. Second, as demonstrated in Figure 3, Followers thresholds can be lower than those of Loner making Follower willing to"test the ground" before Loner enters. This is not the case when λ is close to 1 since high persistence of the state makes "testing the ground" more costly.

Figure 4: The Stochastic-State Model - Numeric Approximation of Value Functions and Simulation of Beliefs and Total Investment when Loner Receives Bad News



Notes: Panel A depicts numeric approximations of Loner and Follower's value functions for the parameters $x = 1$, $\delta = 0.95$, $p = 0.8$ and $\lambda = 0.98$. For Follower, the approximation is of $U(\mu, 0)$, i.e., his value function while Loner is investing. Panels B and C depict the dynamics in a simulation of this model in which both agents have a prior belief of $0.27 < M^L$ and no news arrive until Loner observes π^B in period 8. In panel B solid lines represent agents' beliefs and dashed lines represent thresholds. Note that Follower's threshold is state-dependent (see Figure 3 and associated discussion).

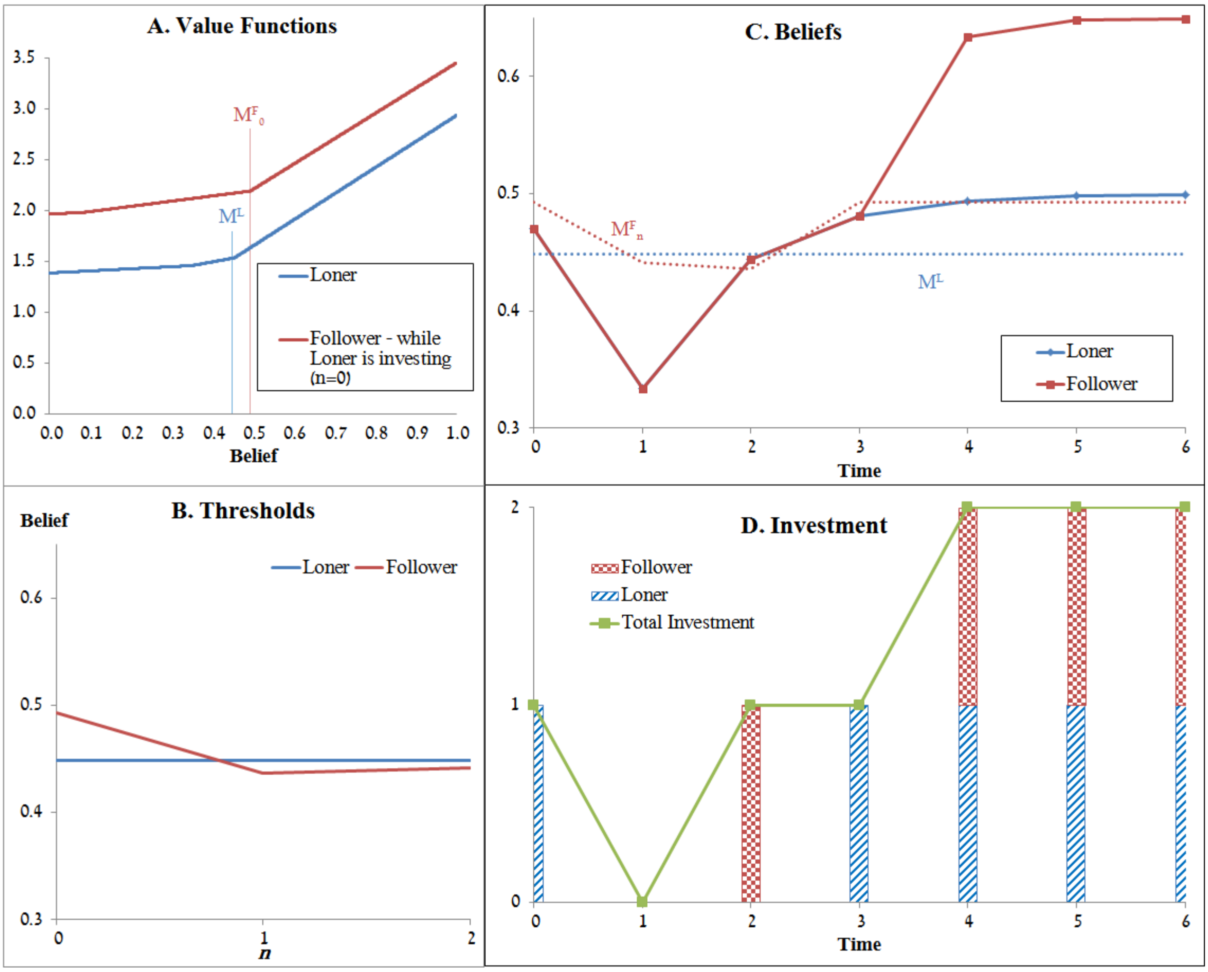
his belief downwards and exit the market. This leads Follower to experience a "Wile E. Coyote moment" and exit as well. Consequently, activity plunges. Note that in period 31, i.e., $\bar{n} = 22$ periods after observing π^B, Loner resumes investment and a new cycle of investment and learning initiates. As in the constant state setting, one can notice that disagreement between agents is more prominent when activity is high and *private* information is generated. In contrast, when activity plunges due to Loner leaving the market, disagreement also vanishes.

4.3 When Follower Chooses to Test the Waters

Proposition 5 established that Followers enters the market after Loner if the state of the world is persistent enough, i.e., when λ is close to one. In this section I demonstrate that if the state is more fickle, it may be that Follower will choose to "test the waters" and enter before Loner in equilibrium.

Panels C and D in Figure 5 depict a simulation of the model with $\lambda = 2/3$, $p = 0.1$, $x = 1$ and $\delta = 0.95$. With these parameters Loner is known to renter the market $\bar{n} = 2$ periods after he exits. The prior belief in the simulation is assumed to be such that Loner enters the market in period 0 and Follower stays out. Furthermore, it is assumed that Loner observes a bad signal in period 0 but afterwards no meaningful signal arrives to the market. Following the arrival of bad news, Loner exits the market in period 1 leading to the update of both agents' beliefs to $1 - \lambda = 1/3$ in period 1. In period 2 both agents update their belief according to the Markovian process to 4/9 which is *above Follower's threshold but below Loner's*. At this point Follower enters the market without observing any reassuring news from Loner. This is because Follower knows that Loner will enter the market in the next period so the cost of "testing the waters" is relatively low. Without observing good news in period 2, Follower is not optimistic enough to stay in the market and he exits in period 3. He renters only after observing Loner staying in the market for two periods.

Figure 5: Numeric Approximation of Value Functions and Simulation of Beliefs and Investment with a Fickle State of the World



Notes: Panel A depicts numeric approximations of Loner and Follower's value functions for the parameters $x = 1$, $\delta = 0.95$, $p = 0.1$ and $\lambda = 2/3$. For Follower, the approximation is of $U(\mu, 0)$, i.e., his value function while Loner is investing. Panel B depicts numeric approximations of agents' thresholds as a function of the number of periods remaining until Loner enters the market, n. Panels C and D depict the dynamics in a simulation of this model in which both agents have a prior belief of $0.47 \in (M^L, M_0^F)$, Loner observes a bad signal in period 0 and afterwards no news arrive to either agent. Panel C depicts the evolution of agents' beliefs (solid lines) and numeric approximations of their thresholds (dashed lines). Panel D depicts active investors in each period.

5 The Public Information Case

The private information setting of previous sections endogenously generates asymmetry in the quality of information embodied in Loner's actions. While an exit decisively indicates that Loner observed a bad signal, staying in the market only indicates that Loner has not yet observed such a signal, meaning that so far he received payoffs of zero and π^G. One might wonder if the difference in information quality is not responsible for some (or all) of the asymmetric dynamics in the model.

For this purpose in this section I compare Follower's behavior to a model with *public information* (I still restrict attention to a setting in which Follower observes Loner but not vice versa). I assume that there are two followers in the market: Follower A that observes Loner's actions as in the baseline model, and Follower B that observes Loner's *signals*. By allowing Follower B to observe Loner's signals I break the asymmetric quality of information that is generated with private information.

As it turns out, this has no qualitative effect on the main results. If λ is close enough to 1, Follower B still enters the market strictly after Loner and exits with him if Loner observes a bad signal. They key insight is that gradual entry is due solely to the free-riding effect which is of course also relevant for Follower B. An abrupt exit occurs when Follower realizes that Loner observed a bad signal and this happens whether he sees Loner receive that signal or just sees him leave the market.

However, public information does affect Follower's incentive to enter and the magnitude of the asymmetry in dynamics in the since that it affects the timing of Follower's entry to the market (after Loner has already entered). As I will show in this section, since Follower B observes information of higher quality when Loner is in the market, his cost from entering and observing adverse news is lower than Follower A's and thus his threshold for entry is also lower. However, if the state is persistent enough this difference is small and does not stimulate more investment in equilibrium. In fact, generally Follower B enters the market after Follower A, i.e., public information defers entry and experimentation. Since the result of abrupt exit is still maintained, this means that public information magnifies the asymmetry in dynamics.

5.1 Follower B

Denote Follower B's value function by $U_B(\mu, n)$. It differs from Follower A's value function only when Loner is in the market and Follower does not observe a meaningful signal (i.e, receives a payoff of zero). This is because when Loner is out of the market both followers are exposed to the same information, and when their signal is informative the information they observe from Loner is redundant. In other words, U_B differs from U in $U_B^N(\mu, 0)$.

Formally, Follower B's value function is defined as follows:

$$U_B(\mu, n) = \max\{U_B^N(\mu, n), U_B^I(\mu, n)\}$$

where,

$$\begin{aligned} U_B^N(\mu,0) =& p\delta U_B\Big((2\lambda-1)\mu+1-\lambda,0\Big)+(1-p)\delta\Big[\mu U_B(\lambda,0)+(1-\mu)U_B(1-\lambda,\bar{n})\Big], \\ U_B^I(\mu,0) =& (2\mu-1)x+pU_B^N(\mu,0)+ \\ & (1-p)\mu\delta U_B(\lambda,0)+(1-p)(1-\mu)\delta\Big[pU_B(1-\lambda,0)+(1-p)U_B(1-\lambda,\bar{n})\Big] \end{aligned}$$

and for $n \in \{1, ..., \bar{n}\}$,

$$\begin{aligned} U_B^N(\mu,n) =& \delta U_B\Big((2\lambda-1)\mu+1-\lambda,n-1\Big), \\ U_B^I(\mu,n) =& (2\mu-1)x+pU_B^N(\mu,n)+(1-p)\delta\Big[\mu U_B\big(\lambda,n-1\big)+(1-\mu)U_B\big(1-\lambda,n-1\big)\Big] \end{aligned}$$

Similar arguments to those given in the proof of Proposition 4 can be used to show that it holds for U_B as well, meaning that Follower B's value function shares the same properties as Follower A's value function. However, because Follower B observes information of higher quality, his value function is higher than Follower A's:

Proposition 8. *$U_B(\mu, n) \geq U(\mu, n)$ for all $n = 0, ..., \bar{n}$ and $\mu \in [0, 1]$.*

5.2 The Effect of Public Information on Follower's Incentives and on Asymmetric Dynamics

First, I will formally show that even though public information eliminates differences in signal quality that arise with private information, it still generates asymmetric dynamics. Note that the results of abrupt versus gradual entry (Proposition 6) and slow recovery (Corollary 7) still hold in the public information setting, i.e., if Follower A is substituted by Follower B. To complete the argument of asymmetric dynamics, the following proposition shows that Follower B also free-rides on Loner's information and enters the market after him when the state of the world is persistent enough:

Proposition 9. ***(Free-Riding and Gradual Entry with Public Information)***
Assume Loner and Follower B share the same belief and Loner is out of the market. If λ is close enough to 1 then Follower B will enter the market strictly after Loner.

Next, Proposition 10 highlights the differences between the two information settings: it compares the two followers' incentives to enter the market and their behavior in equilibrium.

Proposition 10. ***(Incentives and Entry - Public vs. Private Information)***

1. *Denote Follower B's threshold when Loner is in the market (i.e., $n = 0$) by M_0^{FB}, then $M_0^{FB} \leq M_0^F$.*
2. *$M_0^{FB} = M_0^F$ when $\lambda = 1$.*
3. *In equilibrium, if λ is close enough to 1, Follower B enters the market before Follower A if and only if Loner receives a good signal π^G while both followers are out.*

As aforesaid, the free-riding effect holds in both information settings and in equilibrium both followers enter the market after Loner. The question then remains which follower enters first after Loner? The first item in Proposition 10 states that while Loner is in the market Follower B has a (weakly) lower threshold for investment compared to Follower A. This is because Follower B has a smaller cost for entering the market and receiving bad news. Nonetheless, item 3 implies that Follower B will generally enter *after* Follower A:

Corollary 11. ***Public information is more likely to defer entry and experimentation.***
In equilibrium, if λ is close enough to 1, the probability that Follower B will enter before Follower A is smaller than 0.5.

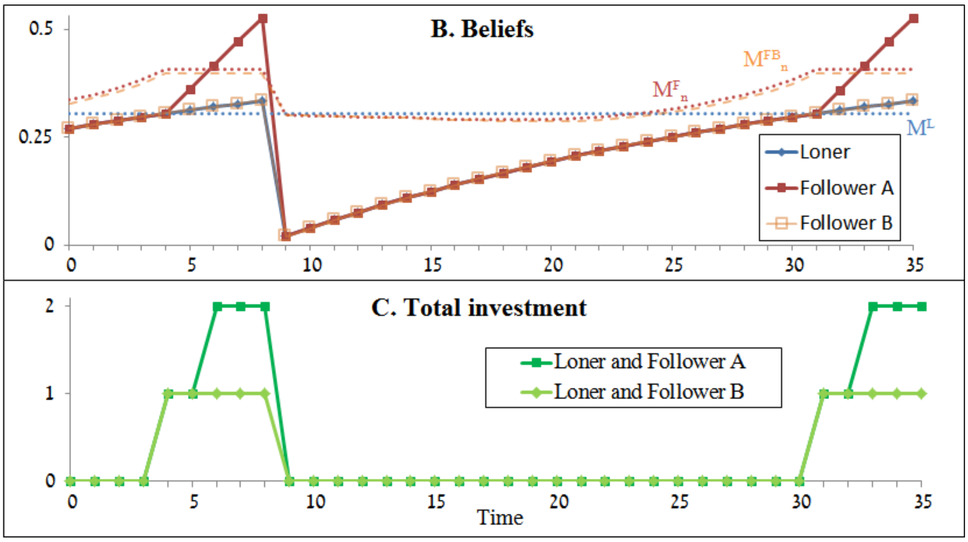
Figure 6 depicts a similar simulation to the one from Figure 4, but this time with the two types of followers. Note that Follower B's threshold is lower than A's but it is quite close. Further note that while Follower B is out of the market, his belief coincides with Loner's. Because Loner doesn't receive any news until period 8 in this example, their beliefs remain relatively low and do not cross Follower B's threshold. Thus, Follower B does not even enter the market. However, Follower A is more optimistic and his belief crosses his threshold after two periods. This example demonstrates the general case in which public information defers entry and experimentation. Follower B would have entered before A only if Loner would have received a good signal in period 4 or 5, while Follower A is still out. But this is highly unlikely:

$$\Pr\left(\pi_t^L = \pi^G \vee \pi_{t+1}^L = \pi^G \middle| \text{Loner entered in } t,\, \theta_t\right) = \begin{cases} (1-\lambda)(1-p) & \text{if } \theta_t = B \\ 1-p+p\lambda(1-p) & \text{if } \theta_t = G \end{cases}$$

Since the state is more likely to be B than G when Loner enters, the total probability is very low (the general proof is given the Appendix):

$$\Pr\left(\pi_t^L = \pi^G \vee \pi_{t+1}^L = \pi^G \middle| \text{Loner entered in t}\right) < 0.5\Big[(1-\lambda)(1-p) + 1 - p + p\lambda(1-p)\Big] = 0.5\Big[0.004 + 0.396\Big] = 0.2$$

Figure 6: Public vs. Private information - Simulation of Beliefs and Total Investment when Loner Receives Bad News



Notes: The figure depicts a simulation of the model with parameters $x = 1$, $\delta = 0.95$, $p = 0.8$ and $\lambda = 0.98$. The simulation includes three agents: Loner, Follower A which observes Loner's actions and Follower B which observes Loner's signals. In the simulation all agents have a prior belief of $0.27 < M^L$ and no news arrive until Loner observes π^B in period 8. Panel B depicts the evolution of beliefs (solid lines) and thresholds (dotted lines) which are state-dependent. Panel C depicts aggregate investment.

6 Evolution of Disagreement over the Business Cycle

As demonstrated in the example of Figure 4, agents' beliefs can vary substantially. Furthermore, this variation is not constant over the business cycle. Specifically, agents' beliefs become perfectly aligned once Loner exits the market: no matter if Follower is in or out at that point, Loner's exit signals that he observed π^B so both agents update their beliefs to $1 - \lambda$ and they remain aligned for the next $\bar{n}$ periods until Loner re-enters the market.

Before elaborating on the dynamics of disagreement, I should formally define what it is. Disagreement is a concept that stands for the dispersion of beliefs. It is therefore naturally captured by the variance in periodic beliefs $Var(\mu_t^i) = 0.25(\mu_t^L - \mu_t^F)^2$. To discuss the cyclicality of this variable I compare it to total investment which is the number of agents in the market and may equal 0, 1 or 2.

The above discussion implies that when total investment is zero then disagreement also equals zero. Of course, if some agent is in the market then he produces private signals which can drive beliefs apart, meaning that disagreement is positive when total investment is positive. It follows that disagreement is expected to be pro-cyclical, i.e., positively correlated with investment.

Panel A in Table 1 shows this correlation in model simulations with different values of p and λ. It is apparent that for most parameter values this correlation is indeed positive. The exceptions are low values of λ combined with high values of p which generate slightly negative correlations. For these values payoffs are extremely uninformative: when λ approaches 0.5 the state is approximately white noise and when p approaches 1 payoffs are almost always 0 and are essentially independent of the state. Thus, in these settings disagreement is always very low and slightly rises only when Follower leaves the market before Loner (i.e., total investment equals 1), causing the small negative correlation.

To understand the role of the information structure in generating these results, Panel B shows the correlation between investment and disagreement in simulations of a similar model

Table 1: Correlation between Activity and Disagreement from Simulations with Different Model Parameters

A. Loner and Follower (Baseline)

p \ λ	0.52	0.6	0.7	0.8	0.9	0.98
0.1	0.23	0.20	0.17	0.13	0.10	0.06
0.2	0.09	0.27	0.23	0.18	0.12	0.07
0.3	0.12	0.30	0.27	0.21	0.14	0.07
0.4	0.13	0.29	0.28	0.22	0.15	0.06
0.5	0.11	0.20	0.28	0.23	0.17	0.07
0.6	0.07	0.18	0.25	0.23	0.19	0.08
0.7	0.00	0.12	0.18	0.24	0.21	0.11
0.8	-0.07	0.06	0.12	0.19	0.22	0.13
0.9	-0.10	-0.05	0.05	0.11	0.20	0.18

B. Two Loners

p \ λ	0.52	0.6	0.7	0.8	0.9	0.98
0.1	0.07	0.06	0.07	0.06	0.06	0.04
0.2	0.06	0.06	0.07	0.08	0.07	0.04
0.3	0.01	0.01	0.06	0.08	0.07	0.04
0.4	-0.05	-0.04	0.04	0.07	0.08	0.04
0.5	-0.14	-0.02	-0.01	0.05	0.07	0.05
0.6	-0.24	-0.09	-0.07	0.01	0.07	0.05
0.7	-0.37	-0.19	-0.08	-0.01	0.05	0.06
0.8	-0.47	-0.27	-0.17	-0.10	0.02	0.07
0.9	-0.59	-0.39	-0.24	-0.20	-0.05	0.08

Color Legend: -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0.0 0.1 0.2 0.3 0.4 0.5 0.6

Note: Each entry in a table shows the correlation between total investment and disagreement $Var(\mu_t^i) = 0.25(\mu_t^F - \mu_t^L)^2$ for different values of p and λ (note that λ is assumed throughout the paper to be larger than 0.5 and since for $\lambda = 1$ a simulation futile I simulate over values of λ in $(0.5, 1)$). Panel A shows results from simulations of the baseline model with Loner and Follower; Panel B shows results from a model with two isolated agents that do not learn from each other. In all simulations: $\delta = 0.95$, $x = 1$, number of observations $= 10^6$.

with two isolated agents that do not observe each other's actions. Comparing the two tables shows that without the additional learning mechanism of observing another agent's actions, the correlation between activity and disagreement falls and might even be very negative. This is true even though the second model also incorporates private signals which are generated only when activity is high. The lower correlation stems from periods of zero activity since any disagreement that was formulated when one agent left the market before the other is maintained as long as they are both out. As aforesaid, this is not the case in the baseline model in which zero activity necessarily means that beliefs are aligned.

7 Conclusion

The phenomenon of slow booms and sudden crashes is prevalent in many economic indicators of activity and beliefs (e.g., expectations). This paper showed that such asymmetries arise in a setting in which actions generate information and agents learn from each other. The mechanism was studied for a simple setting of two agents: an isolated agent (Loner) and an observer (Follower). In an otherwise completely symmetric setting, I showed that agents enter the market gradually but tend to exit simultaneously. Since Follower "free-rides" on the information generated by Loner, he prefers to observe him stay in the market for a while before entering himself. This is the cause for the gradual increase in activity. On the other hand, Follower acts immediately upon observing Loner exit the market, as this action decisively indicates that Loner observed bad news.

In essence, this mechanism is independent of the quality of information observed by Follower, i.e., Loner's actions or signals. However, if Follower observes Loner's signals and not just is his actions, his threshold for entry is lower as his access to superior information makes it less costly for him to enter the market. Nonetheless, the fact that he can tell if Loner is in the market because he observed good news or because he had not yet observed bad news, on average makes him wait out of the market longer than when he observes Loner's actions.

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Appendix A Proofs

A.1 Proof of Proposition 3 (Properties of Loner's Value Function)

1. Define the mappings:

$$T_L(f)(\mu) = \max\left\{ \delta f\Big((2\lambda-1)\mu+1-\lambda\Big),\right.$$
$$\left.(2\mu-1)x + p\delta f\Big((2\lambda-1)\mu+1-\lambda\Big) + \delta(1-p)[\mu V(\lambda)+(1-\mu)V(1-\lambda)]\right\}$$

   It is easy to verify that T_L satisfies Blackwell's conditions for a contraction mapping and preserves monotonicity, continuity and convexity. It follows that this mapping has a unique fixed point, V, that is non-decreasing, continuous and convex.[6]

2. Since $V \geq 0$ it is clear that Loner invests when $\mu \geq 0.5$. Thus there exists a threshold $M^L \leq 0.5$ such that $V^N(M^L) = V^I(M^L)$ and Loner invests in $[M^L, 1]$. Furthermore, note that if $\mu \in [M^L, 1]$ then also $(2\lambda-1)\mu+1-\lambda \in [M^L, 1]$ (if $\mu \geq 0.5$ then $(2\lambda-1)\mu+1-\lambda \geq 0.5$, and if $M^L \leq \mu \leq 0.5$ then $\mu < (2\lambda-1)\mu+1-\lambda$). This implies that $\forall \mu \in [M^L, 1]$,

$$V^I(\mu) = (2\mu-1)x + p\delta V^I\Big((2\lambda-1)\mu+1-\lambda\Big) + \delta(1-p)\Big[\mu V(\lambda)+(1-\mu)V(1-\lambda)\Big]$$

   It can now be verified that in this segment $V(\mu) = \alpha^L\mu + \beta^L$ for α^L and β^L as specified in the proposition, and that $M^L = \frac{x-(1-p)[V(0)-\beta^L]}{(1+p)x-(1-p)[V(0)-\beta^L]}$.

   Note that it must be that $M^L > 1-\lambda$. Otherwise, if $M^L \leq 1-\lambda$, then $V(\mu) = \alpha^L\mu + \beta^L$ for all $\mu \geq 1-\lambda$ which implies that

$$V(\mu) = \max\left\{\delta\left[\alpha^F\Big((2\lambda-1)\mu+1-\lambda\Big)+\beta^F\right],\quad (2\mu-1)x+\delta\left[\alpha^F\Big((2\lambda-1)\mu+1-\lambda\Big)+\beta^F\right]\right\}$$

   and thus $M^L = 0.5$ but this is a contradiction since $0.5 > 1-\lambda$.

   It remains to show that M^L is the unique threshold for investment. Let $IG^L(\mu) = V^I(\mu) - V^N(\mu)$ denote Loner's gain from investment. I will show that IG^L is increasing which will imply that M^L is the unique belief that nullifies IG^L.

$$IG^L(\mu) = (2\mu-1)x + (1-p)\delta\Big[\mu V(\lambda)+(1-\mu)V(1-\lambda) - V\Big((2\lambda-1)\mu+1-\lambda\Big)\Big] =$$
$$[(1+p)\mu-1]x + (1-p)\Big[\mu V(1)+(1-\mu)V(0) - \delta V\Big((2\lambda-1)\mu+1-\lambda\Big)\Big]$$

---

[6] Blackwell's sufficient condition for a contraction: Let $X \subseteq \mathbb{R}^n$ and let B be a space of bounded functions $f: X \to \mathbb{R}$. Let $T: B \to B$ be a mapping that satisfies: (1) Monotonicity: if $f \leq g$ then $Tf \leq Tg$. (2) Discounting: there exists $\delta \in (0,1)$ such that $T[(f+a)](x) \leq Tf(x) + \delta a$ for all $f \in B, a \geq 0, x \in X$. Then T is a contraction mapping.
If T satisfies these conditions and B is a complete space then following Banach's fixed point theorem, T has a unique fixed point V in B. Furthermore, $\lim_{n\to\infty} T^n(f) = V$ for every $f \in B$.
In the context of Proposition 3, B is the space of bounded, non-decreasing, continuous and convex functions. Thus V's existence and its properties follow from the fact that T_L is a contraction mapping and $T_L(B) \subseteq B$.

$$IG^L(\mu+\Delta) - IG^L(\mu) = (1+p)x\Delta + (1-p)\Big[V(1) - V(0)\Big]\Delta -$$
$$(1-p)\delta\left[V\Big((2\lambda-1)(\mu+\Delta)+1-\lambda\Big) - V\Big((2\lambda-1)\mu+1-\lambda\Big)\right] \geq$$
$$(1+p)x\Delta + (1-p)\Big[V(1) - V(0)\Big]\Delta - (1-p)\delta\alpha^L(2\lambda-1)\Delta =$$
$$(1+p)x\Delta + (1-p)\Big[\alpha^L + \beta^L - V(0)\Big]\Delta - (1-p)\delta\alpha^L(2\lambda-1)\Delta =$$
$$(1+p)x\Delta + (1-p)\alpha^L\Big[1-\delta(2\lambda-1)\Big]\Delta - (1-p)\Big[V(0)-\beta^L\Big]\Delta$$

where the inequality in the second line is due to $V's$ convexity which implies that the maximal slope of V is α^L. Now, since $M^L = \frac{x-(1-p)[V(0)-\beta^L]}{(1+p)x-(1-p)[V(0)-\beta^L]} \in (0, 0.5]$ it follows that $(1-p)[V(0)-\beta^L] \leq x$, so:

$$\big[IG^L(\mu+\Delta) - IG^L(\mu)\big]/\Delta \geq (1-p)\alpha^L\Big[1-\delta(2\lambda-1)\Big] > 0$$

A.2 Proof of Proposition 4 (Properties of Follower's Value Function)

Before turning to the proof of Proposition 4, I will prove two lemmas. Lemma 12 shows that $U(\mu, 0)$, Follower's value function when Loner is in the market, can be written as the sum of a "self contained" value function W and a linear function. By "self contained" I mean that W is only a function of μ, as opposed to $U(\mu, 0)$ that also depends on $U(\mu, \bar{n})$. In other words, a Follower that maximizes according to W does not take into account that Loner may exit the market. The purpose of this lemma is to facilitate the proofs of some of $U(\mu, 0)$'s properties - those that are preserved when adding a linear function such as convexity and the threshold for investment. Lemma 13 is used to prove W's convexity.

Lemma 12. *$U(\mu, 0)$ can be written as the sum of a linear function and a function W as follows:*

$$U(\mu, 0) = W(\mu) + a\mu + b \tag{6}$$

where,

$$W(\mu) = \max\{W^N(\mu), W^I(\mu)\},$$
$$W^N(\mu) = [p+(1-p)\mu]\delta W\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu} + 1 - \lambda\right) + (1-p)(1-\mu)\delta W(1-\lambda),$$
$$W^I(\mu) = (2\mu-1)x + pW^N(\mu) + \delta(1-p)\Big[\mu W(\lambda) + (1-\mu)W(1-\lambda)\Big] \text{ and}$$
$$[a,b] = \frac{(1-p)[W(0) - U(0,\bar{n})]}{(1-\delta\lambda)(1-\delta p) + \delta p(1-\delta)(1-\lambda)}\Big[1-\delta,\ -(1-\delta\lambda)\Big]$$

Proof. Denote $f(\mu) = U(\mu, 0) - W(\mu)$, thus:

$$U(\mu,0) = W(\mu)+f(\mu) = \max\left\{U^N(\mu)+\Big[W^N(\mu)-U^N(\mu,0)+f(\mu)\Big],\ U^I(\mu,0)+\Big[W^I(\mu)-U^I(\mu)+f(\mu)\Big]\right\}$$

So necessarily $f(\mu) = U^N(\mu, 0) - W^N(\mu) = U^I(\mu, 0) - W^I(\mu)$. To find f I take an "educated guess" that it is linear and solve the two equations:

$$a\mu + b = U^N(\mu, 0) - W^N(\mu) =$$
$$[p+(1-p)\mu]\delta\left[a(2\lambda-1)\frac{\mu}{p+(1-p)\mu} + a(1-\lambda) + b\right] + (1-p)(1-\mu)\delta\big[U(1-\lambda,\bar{n}) - W(1-\lambda)\big] =$$
$$\delta\Big[a(2\lambda-1)\mu + [p+(1-p)\mu][a(1-\lambda)+b]\Big] + (1-p)(1-\mu)\delta\big[U(1-\lambda,\bar{n}) - W(1-\lambda)\big] =$$

$$a\mu + b = U^I(\mu, 0) - W^I(\mu) =$$

$$p[a\mu+b]+\delta(1-p)\Big[\mu(a\lambda+b)+(1-\mu)\Big[pU(1-\lambda,\bar{n})-W(1-\lambda)\Big]\Big]+(1-p)^2(1-\mu)\delta U(1-\lambda,\bar{n}) =$$

$$p[a\mu+b]+\delta(1-p)\Big[\mu(a\lambda+b)+(1-\mu)p\Big[a(1-\lambda)+b\Big]\Big]+(1-p)^2(1-\mu)\delta\big[U(1-\lambda,\bar{n})-W(1-\lambda)\big]$$

Both equations are equivalent to:

$$a\mu + b = \delta\Big[\mu(a\lambda + b) + (1-\mu)p\Big[a(1-\lambda) + b\Big]\Big] + (1-p)(1-\mu)\big[U(0,\bar{n}) - W(0)\big]$$

which yields the solution:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1-\delta\lambda & 1-\delta \\ -\delta p(1-\lambda) & 1-\delta p \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ (1-p)[U(0,\bar{n})-W(0)] \end{bmatrix} =$$

$$\frac{(1-p)[U(0,\bar{n})-W(0)]}{(1-\delta\lambda)(1-\delta p)+\delta p(1-\delta)(1-\lambda)} \begin{bmatrix} -(1-\delta) \\ 1-\delta\lambda \end{bmatrix}$$

□

Lemma 13. *Let $f : A \to \mathbb{R}_+$ be a real function such that $0 \in A \subseteq \mathbb{R}$. Define $g : A \to \mathbb{R}$ as follows:*

$$g(x; f) = [p + (1-p)x] f\left((2\lambda - 1)\frac{x}{p + (1-p)x} + 1 - \lambda\right)$$

If f is increasing and convex (concave) then g is also increasing and convex (concave).

Proof. Let $0 \le x < y \le 1$ and $c \in (0,1)$. From the convexity of f it follows that,

$$f\left((2\lambda-1)\frac{cx+(1-c)y}{p+(1-p)[cx+(1-c)y]}+1-\lambda\right) \le$$
$$f\left((2\lambda-1)\frac{x}{p+(1-p)x}+1-\lambda\right) + \frac{(1-c)p[f(y)-f(x)](2\lambda-1)(y-x)}{[p+(1-p)x][p+(1-p)[cx+(1-c)y]]}$$

and that,

$$f\left((2\lambda-1)\frac{cx+(1-c)y}{p+(1-p)[cx+(1-c)y]}+1-\lambda\right) \le$$
$$f\left((2\lambda-1)\frac{y}{p+(1-p)y}+1-\lambda\right) - \frac{cp[f(y)-f(x)](2\lambda-1)(y-x)}{[p+(1-p)y][p+(1-p)[cx+(1-c)y]]}$$

Now,

$$g(cx+(1-c)y; f) = \Big[p+(1-p)[cx+(1-c)y]\Big] f\left((2\lambda-1)\frac{cx+(1-c)y}{p+(1-p)[cx+(1-c)y]}+1-\lambda\right) =$$
$$c[p+(1-p)x] f\left((2\lambda-1)\frac{cx+(1-c)y}{p+(1-p)[cx+(1-c)y]}+1-\lambda\right) +$$
$$(1-c)[p+(1-p)y] f\left((2\lambda-1)\frac{cx+(1-c)y}{p+(1-p)[cx+(1-c)y]}+1-\lambda\right) \le$$
$$c[p+(1-p)x] f\left((2\lambda-1)\frac{x}{p+(1-p)x}+1-\lambda\right) + c\frac{(1-c)p[f(y)-f(x)](2\lambda-1)(y-x)}{p+(1-p)[cx+(1-c)y]} +$$
$$(1-c)[p+(1-p)y] f\left((2\lambda-1)\frac{y}{p+(1-p)y}+1-\lambda\right) - (1-c)\frac{cp[f(y)-f(x)](2\lambda-1)(y-x)}{p+(1-p)[cx+(1-c)y]} =$$
$$cg(x; f) + (1-c)g(y; f)$$

□

I now turn to the proof of Proposition 4.

Proposition 4.

1. There exists a unique value function $U : [0,1] \times \{0, ..., \bar{n}\} \to \mathbb{R}_+$ that satisfies (5). This function is non-decreasing, continuous and convex in the first argument.

2. $U(\mu, n) \geq V(\mu)$ for all $n = 0, ..., \bar{n}$ and $\mu \in [0,1]$.

3. For all $n = 0, ..., \bar{n}$ there exists a unique threshold belief $M_n^F \in (1-\lambda, 0.5)$ such that Follower invests in state (μ, n) if and only if $\mu \geq M_n^F$, and $U(\mu, n)$ is TLC .

4. Define $U(\mu, n)$ as in (5) for all $n \in \mathbb{N}$ (i.e, extend the definition for $\bar{n}+1, \bar{n}+2, ...$) then $\lim_{n\to\infty} U(\mu, n) = V(\mu)$. This means that as Loner's entry is further away, Follower acts as an isolated agent.

Proof.

1. The set of states Follower may be facing is $S = [0,1] \times \{0, ..., \bar{n}\}$. Denote by $Q^a(s, s')$ the probability of transition from state s' to state s, depending on Followers action a.

   If Follower chooses not to invest ($a = N$):

$$Q^N\Big(s, (\mu, 0)\Big) = \begin{cases} p + (1-p)\mu & \text{if } s = \Big((2\lambda - 1)\frac{\mu}{p+(1-p)\mu} + 1 - \lambda, 0\Big) \\ (1-p)(1-\mu) & \text{if } s = (1-\lambda, \bar{n}) \\ 0 & \text{otherwise} \end{cases}$$

   For $n \in \{1, ..., \bar{n}\}$, $$Q^N\Big(s, (\mu, n)\Big) = \begin{cases} 1 & \text{if } s = \Big((2\lambda - 1)\mu + 1 - \lambda, n-1\Big) \\ 0 & \text{otherwise} \end{cases}$$

   If Follower chooses to invest ($a = I$):

$$Q^I\Big(s, (\mu, 0)\Big) = \begin{cases} p[p + (1-p)\mu] & \text{if } s = \Big((2\lambda - 1)\frac{\mu}{p+(1-p)\mu} + 1 - \lambda, 0\Big) \\ (1-p)\mu & \text{if } s = (\lambda, 0) \\ (1-p)p(1-\mu) & \text{if } s = (1-\lambda, 0) \\ (1-p)(1-\mu) & \text{if } s = (1-\lambda, \bar{n}) \\ 0 & \text{otherwise} \end{cases}$$

   For $n \in \{1, ..., \bar{n}\}$, $$Q^I\Big(s, (\mu, n)\Big) = \begin{cases} p & \text{if } s = \Big((2\lambda - 1)\mu + 1 - \lambda, n-1\Big) \\ (1-p)\mu & \text{if } s = (\lambda, n-1) \\ (1-p)(1-\mu) & \text{if } s = (1-\lambda, n-1) \\ 0 & \text{otherwise} \end{cases}$$

   Follower's dynamic choice problem (5) can then be written as follows:

$$U(\mu, n) = \max\left\{\delta \sum_{s\in S} U(s) Q^N\Big(s, (\mu, n)\Big),\ (2\mu - 1)x + \delta \sum_{s\in S} U(s) Q^I\Big(s, (\mu, n)\Big)\right\}$$

   Now, define the mapping:

$$T_F(f)(\mu, n) = \max\left\{\delta \sum_{s\in S} f(s) Q^N\Big(s, (\mu, n)\Big),\ (2\mu - 1)x + \delta \sum_{s\in S} f(s) Q^I\Big(s, (\mu, n)\Big)\right\}$$

It is easy to verify that T_F satisfies Blackwell's condition for a contraction mapping and preserves monotonicity and continuity in μ. It follows that this mapping has a unique fixed point, $U(\mu,n)$, which is non-decreasing and continuous in μ.

To show that $U(\mu,0)$ is convex in μ it suffices to show that $W(\mu)$ from Lemma 12 is convex. Following Lemma 13, $W(\mu)$ is a maximum of two convex functions and thus convex. Now, from the recursive structure of $U(\mu,n)$ for $n=1,...,\bar{n}$ it is clear that these functions are also convex in μ.

2. Following Equation (5), note that for all $n=1,...,\bar{n}$ we can write $U(\mu,n)=T_LU(\mu,n-1)$ where T_L is the mapping defined in Proposition 3 whose fixed point is V. Further note that the space $B\equiv\Big\{f|f:[0,1]\times\{0,...,\bar{n}\}\to\mathbb{R}\Big\}$ is homeomorphic to the space $\tilde{B}\equiv\Big\{\tilde{f}|\tilde{f}:[0,1]\to\mathbb{R}^{\bar{n}+1}\Big\}$. Thus, T_F that is defined on B has an equivalent transformation $\tilde{T}_F$ that operates on $\tilde{B}$ and its fixed point is $\tilde{U}(\mu)\equiv\big(U(\mu,0),...,U(\mu,\bar{n})\big)'$:

$$\tilde{T}_F(\tilde{f})(\mu)=\tilde{T}_F\begin{pmatrix}f(\mu,0)\\ f(\mu,1)\\ \vdots\\ f(\mu,\bar{n})\end{pmatrix}\equiv\begin{pmatrix}T_F(f)(\mu,0)\\ T_F(f)(\mu,1)\\ \vdots\\ T_F(f)(\mu,\bar{n})\end{pmatrix}=$$
$$\begin{pmatrix}\max\Big\{\delta\sum_{s\in S}f(s)Q^N\Big(s,(\mu,0)\Big),\,(2\mu-1)x+\delta\sum_{s\in S}f(s)Q^I\Big(s,(\mu,0)\Big)\Big\}\\ T_L(f)(\mu,0)\\ \vdots\\ T_L(f)(\mu,\bar{n}-1)\end{pmatrix}$$

Since $\tilde{U}$ is $\tilde{T}_F$'s fixed point, in order to show that it is greater than $\tilde{V}\equiv\big(V,...,V\big)'$, it suffices to show that if $\tilde{f}(\mu)=(f(\mu,0),...,f(\mu,\bar{n}))'\geq\tilde{V}(\mu)$ then $\tilde{T}_F(\tilde{f})(\mu)\geq\tilde{V}(\mu)$ as well.[7]

Since T_L is monotone then $T_L(f)(\mu,n)\geq T_L(V)(\mu)=V(\mu)$ for all $n=0,...,\bar{n}-1$. As for the first argument of $\tilde{T}_F(\tilde{f})(\mu)$, it is the maximum of two arguments:

$$\begin{aligned}&\delta\sum_{s\in S}f(s)Q^N\Big(s,(\mu,0)\Big)=\\ &[p+(1-p)\mu]\delta f\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda,0\right)+(1-p)(1-\mu)\delta f(1-\lambda,\bar{n})\geq\\ &[p+(1-p)\mu]\delta V\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda\right)+(1-p)(1-\mu)\delta V(1-\lambda)\underbrace{\geq}_{V\,convex}\\ &\delta V((2\lambda-1)\mu+1-\lambda)=V^N(\mu)\end{aligned}$$

and,

$$\begin{aligned}&(2\mu-1)x+\delta\sum_{s\in S}f(s)Q^I\Big(s,(\mu,0)\Big)=\\ &(2\mu-1)x+p\left[\delta\sum_{s\in S}f(s)Q^N\Big(s,(\mu,0)\Big)\right]+(1-p)\mu\delta f(\lambda,0)+\\ &(1-p)(1-\mu)\delta\Big[pf(1-\lambda,0)+(1-p)f(1-\lambda,\bar{n})\Big]\geq\\ &(2\mu-1)x+pV^N(\mu)+(1-p)\mu\delta V(\lambda)+(1-p)(1-\mu)\delta V(1-\lambda)=V^I(\mu)\end{aligned}$$

[7] If $\tilde{f}\geq\tilde{V}$ implies that $\tilde{T}_F(\tilde{f})\geq\tilde{V}$ then $\tilde{T}_F^n(\tilde{f})\geq\tilde{V}$ for all n and in the limit $\tilde{U}=\lim_{n\to\infty}\tilde{T}_F^n(\tilde{f})\geq\tilde{V}$.

Thus, the first argument of $\tilde{T}_F(\tilde{f})(\mu)$ is greater than $\max\{V^N(\mu), V^I(\mu)\} = V(\mu)$.

3. I will prove the claim by induction on n.

   For $n = 0$:

   Following Lemma 12 it suffices to prove the claim for W (since $U(\mu, 0) = W(\mu) + a\mu + b$, the unique threshold of W will also be the unique threshold of $U(\mu, 0)$). The existence of a threshold for W follows from similar arguments to those given in the proof of Proposition 3, i.e., it can be shown that there exists $M_0^F \in (1-\lambda, 0.5)$ such that $W^I(\mu) \geq W^N(\mu)$ for all $\mu \in [M_0^F, 1]$ and that W is linear in this segment. Specifically, it can be shown that

$$W(\mu) = \max\left\{[p+(1-p)\mu]\delta W\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda\right)+(1-p)(1-\mu)\delta W(1-\lambda),\ \alpha^F\mu+\beta^F\right\}$$

   where $\alpha^F = \frac{x(2-\delta-\delta p^2)-(1-\delta)(1-p^2)W(0)}{(1-\delta\lambda)(1-\delta p^2)+\delta p^2(1-\delta)(1-\lambda)}$, $\beta^F = \frac{x-(1-\delta\lambda)\alpha^F}{1-\delta}$ and $M_0^F = \frac{x-(1-p)[W(0)-\beta^F]}{(1+p)x-(1-p)[W(0)-\beta^F]}$.

   It remains to show that M_0^F is the unique threshold. Denote $IG^F(\mu, 0) = W^I(\mu) - W^N(\mu)$. I will show that IG^F is increasing in μ which will imply that M_0^F is the unique belief that nullifies it. Since W is convex, α^F is the maximal slope of any arch of W. Thus,

$$\begin{aligned}
&[W^N(\mu+\Delta) - W^N(\mu)]/\delta = \\
&\quad [p+(1-p)(\mu+\Delta)]W\left((2\lambda-1)\frac{\mu+\Delta}{p+(1-p)(\mu+\Delta)}+1-\lambda\right) - \\
&\quad \Big[p+(1-p)\mu\Big]W\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda\right) - (1-p)\Delta W(1-\lambda) = \\
&\quad (1-p)\Delta\left[W\left((2\lambda-1)\frac{\mu+\Delta}{p+(1-p)(\mu+\Delta)}+1-\lambda\right) - W(1-\lambda)\right] + \\
&\Big[p+(1-p)\mu\Big]\left[W\left((2\lambda-1)\frac{\mu+\Delta}{p+(1-p)(\mu+\Delta)}+1-\lambda\right) - W\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda\right)\right] \leq \\
&(1-p)\Delta\alpha^F(2\lambda-1)\frac{\mu+\Delta}{p+(1-p)(\mu+\Delta)} + \alpha^F(2\lambda-1)\frac{p\Delta}{p+(1-p)(\mu+\Delta)} = \alpha^F(2\lambda-1)\Delta
\end{aligned}$$

   Now,

$$\begin{aligned}
&IG^F(\mu+\Delta, 0) - IG^F(\mu, 0) = (1+p)x\Delta + (1-p)\Big[W(1)-W(0)\Big]\Delta - (1-p)\Big[W^N(\mu+\Delta)-W^N(\mu)\Big] \geq \\
&\quad (1+p)x\Delta + (1-p)\Big[\alpha^F + \beta^F - W(0)\Big]\Delta - (1-p)\delta\alpha^F(2\lambda-1)\Delta \geq \\
&\quad\quad (1-p)\alpha^F\Delta - (1-p)\delta\alpha^F(2\lambda-1)\Delta = (1-p)\alpha^F\Delta\Big[1-\delta(2\lambda-1)\Big] \geq 0
\end{aligned}$$

   where the penultimate inequality is due to fact that $M_0^F < 0.5$ so $W(0) - \beta^F \leq (1+p)x$.

   For $n = 1, ..., \bar{n}$:

   I will prove that $U(\mu, n)$ has a unique threshold for investment $M_n^F \in (1-\lambda, 0.5)$ and that

$U(\mu, n)$ is TLC . Recall that:

$$U(\mu,n) = \max\left\{U^N(\mu,n), U^I(\mu,n)\right\} =$$

$$\max\left\{ \delta U\Big((2\lambda-1)\mu+1-\lambda, n-1\Big), \right.$$

$$\left.(2\mu-1)x+p\delta U\Big((2\lambda-1)\mu+1-\lambda,n-1\Big)+\delta(1-p)[\mu U(\lambda,n-1)+(1-\mu)U(1-\lambda,n-1)]\right\}$$

Since $U(\mu, n-1)$ is convex in μ then for $\mu \geq 0.5$,

$$(1-p)\delta U\Big((2\lambda-1)\mu+1-\lambda,n-1\Big) \leq (2\mu-1)x+\delta(1-p)\Big[\mu U(\lambda,n-1)+(1-\mu)U(1-\lambda,n-1)\Big]$$

This means that there exists $M_n^F < 0.5$ such that Follower invests in (μ, n) for $\mu \geq M_n^F$. The uniqueness of M_n^F follows from the fact that $U(\mu, n-1)$ is convex, using similar arguments to those given in the proof of the uniqueness of M^L (Proposition 3).

It remains to show that $U(\mu, n)$ is TLC. For $\mu \geq M_n^F$,

$$U(\mu,n) = (2\mu-1)x+p\delta U\Big((2\lambda-1)\mu+1-\lambda,n-1\Big)+\delta(1-p)\Big[\mu U(\lambda,n-1)+(1-\mu)U(1-\lambda,n-1)\Big]$$

Since $U(\mu, n-1)$ is TLC with some threshold M then $U(\mu, n)$ is TLC with threshold $\max\left\{M_n^F, \frac{M-(1-\lambda)}{2\lambda-1}\right\}$. In fact, by induction we get that $U(\mu, n)$ is TLC with threshold $\max\left\{M_n^F, \frac{M_{n-1}^F-(1-\lambda)}{2\lambda-1}, ..., 0.5 - \frac{0.5-M_0^F}{(2\lambda-1)^n}\right\}$.

4. As aforesaid, for all $n = 1, ..., \bar{n}$ we can write

$$U(\mu,n) = T_L U(\mu,n-1) = (T_L)^n U(\mu,0) \tag{7}$$

where T_L is the mapping whose fixed point is V. This means that if we extend definition (7) to all $n \in \mathbb{N}$ (holding $U(0, n)$ fixed) then $\lim_{n\to\infty} U(\mu, n) = \lim_{n\to\infty} (T_L)^n U(\mu, 0) = V(\mu)$.[8]

□

A.3 Proof of Proposition 5 (Free-Riding and Gradual Entry)

Unfortunately, in the stochastic state setting agents' thresholds cannot be derived analytically. The main difficulty is that an agent's value in $\mu = 0$ is not derived analytically since it depends on the number of periods until the agent resumes investment which in turn depends on the value at $\mu = 0$. In the constant state model this was not an issue since the agent never returned to the market after leaving so the value at $\mu = 0$ was zero. Therefore, the proof of free-riding in the stochastic case is based on an approximation of thresholds to the static state thresholds. Specifically, Lemma 14 shows that agents' thresholds are continuous in λ at $\lambda = 1$. Thus we can use the analytical solutions of agents' thresholds at $\lambda = 1$ to show that the free-riding effect arises at the proximity of $\lambda = 1$. Lemmas 15 and 16 set bounds on the thresholds to facilitate the proof.

Lemma 14. *Think of Loner's threshold an Follower's threshold when Loner is in the market as functions of λ and denote them by $M^L(\lambda)$ and $M_0^F(\lambda)$, respectively. These thresholds are continuous in λ at $\lambda = 1$.*

[8]Banach's fixed point theorem guarantees that $T_L^n(f) \to V$ for any f. See Footnote 6.

Proof. **Loner's threshold:** Consider the value of V at a belief μ as a function of λ as well and denote it by $V(\mu;\lambda)$. $V(\mu;\lambda)$ is convex in μ and bounded by the present value of a constant payment x. It follows that

$$V(0;\lambda)=\delta V(1-\lambda;\lambda)\leq(1-\lambda)\delta V(1;\lambda)+\lambda\delta W(0;\lambda)\leq\delta(1-\lambda)\frac{x}{1-\delta}+\lambda\delta V(0;\lambda)$$

Thus, $0\leq V(0;\lambda)\leq\frac{\delta(1-\lambda)x}{(1-\delta\lambda)(1-\delta)}$ for all λ and equality holds for $\lambda=1$. It follows that $V(0;\lambda)$ is continuous in λ at $(0;1)$ and its value is zero at this point. Consequently, the following expression is also continuous at $\lambda=1$:

$$D^L(\lambda)\equiv V(0;\lambda)-\beta^L=V(0;\lambda)-\frac{x-(1-\delta\lambda)\alpha^L}{1-\delta}=$$
$$V(0;\lambda)\left[1-\frac{(1-\delta\lambda)(1-p)}{(1-\delta\lambda)(1-\delta p)+\delta p(1-\delta)(1-\lambda)}\right]+\frac{1-\delta\lambda-\delta p(1-\lambda)}{(1-\delta\lambda)(1-\delta p)+\delta p(1-\delta)(1-\lambda)}x \quad (8)$$

Finally, it follows that $M^L(\lambda)=\frac{x-(1-p)D^L(\lambda)}{(1+p)x-(1-p)D^L(\lambda)}$ is continuous at $\lambda=1$.

Follower's threshold: Using similar arguments it can be shown that $W(0;\lambda)$ is continuous at $\lambda=1$ which implies that $D^F(\lambda)\equiv W(0,\lambda)-\beta^F$ and $M_0^F(\lambda)=\frac{x-(1-p)D^F(\lambda)}{(1+p)x-(1-p)D^F(\lambda)}$ are also continuous at that point. □

Lemma 15. *Denote $D^n=U(0,n)-\beta^n$ and $y^n=\frac{x-(1-p)D^n}{(1+p)x-(1-p)D^n}$ for all $n=0,...,\bar{n}+1$.*

1. *$y_0=M_0^F$.*
2. *y^n is the belief that solves $\alpha^n y^n+\beta^n=\delta\left(\alpha^{n-1}[(2\lambda-1)y^n+1-\lambda]+\beta^{n-1}\right)$ and it can be interpreted as follows. If Follower knew that if he would observe a signal 0 in all n consecutive periods he will be investing in all of them (and thus will be on the linear part of U in all the following n periods), then y^n is the belief that makes him indifferent between investing and not investing today.*
3. *$M_n^F\geq y^n$.*

Proof. 1. follows trivially from the solutions of M^F. 2. follows from the recursive structure of α^n and β^n:[9]

$$\begin{pmatrix}\alpha^n\\ \beta^n\end{pmatrix}=\delta\begin{pmatrix}\lambda-p(1-\lambda) & 1-p\\ p(1-\lambda) & p\end{pmatrix}\begin{pmatrix}\alpha^{n-1}\\ \beta^{n-1}\end{pmatrix}+\begin{pmatrix}2x-(1-p)U(0,n)\\ -x+(1-p)U(0,n)\end{pmatrix} \quad (9)$$

[9]Since $U(\mu,n)=U^I(\mu,n)$ is linear on $[0.5,1]$ for all n, the recursive structure can be achieved by solving $\alpha^n\mu+\beta^n=U^I(\mu,n)=(2\mu-1)x+p\delta\left(\alpha^{n-1}[(2\lambda-1)\mu+1-\lambda]+\beta^{n-1}\right)+(1-p)\delta\left[\mu(\alpha^n+\beta^n)+(1-\mu)U(1-\lambda,n-1)\right]$ for $\mu=0.5,1$.

To prove 3. I need to show that $U^I(y^n,n) \leq U^N(y^n,n)$:

$$U^I(y^n,n) = (2y^n-1)x + pU^N(y^n,n) + (1-p)\delta\Big[y^n U(\lambda,n-1) + (1-y^n)U(1-\lambda,n-1)\Big] =$$
$$(2y^n-1)x + p\delta\Big[\alpha^{n-1}\Big((2\lambda-1)y^n+1-\lambda\Big)+\beta^{n-1}\Big]+$$
$$(1-p)\delta\Big[y^n U(\lambda,n-1)+(1-y^n)U(1-\lambda,n-1)\Big]+$$
$$pU^N(y^n,n) - p\delta\Big[\alpha^{n-1}\Big((2\lambda-1)y^n+1-\lambda\Big)+\beta^{n-1}\Big] \underbrace{=}_{Eq.\,(9)}$$
$$\alpha^n y^n + \beta^n +$$
$$pU^N(y^n,n) - p\delta\Big[\alpha^{n-1}\Big((2\lambda-1)y^n+1-\lambda\Big)+\beta^{n-1}\Big] \underbrace{=}_{Lemma\,15.2}$$
$$\delta\Big[\alpha^{n-1}\Big((2\lambda-1)y^n+1-\lambda\Big)+\beta^{n-1}\Big]+$$
$$pU^N(y^n,n) - p\delta\Big[\alpha^{n-1}\Big((2\lambda-1)y^n+1-\lambda\Big)+\beta^{n-1}\Big] =$$
$$pU^N(y^n,n) + (1-p)\delta\Big[\alpha^{n-1}\Big((2\lambda-1)y^n+1-\lambda\Big)+\beta^{n-1}\Big] \leq$$
$$pU^N(y^n,n) + (1-p)\delta U\Big((2\lambda-1)y^n+1-\lambda, n-1\Big) = U^N(y^n,n)$$

□

Lemma 16. $M^L(\lambda) \leq \frac{(1-\delta\lambda)(1-\delta)+\delta(1-\lambda)(2-\delta-p)}{(2-\delta-\delta p)[1-\delta(2\lambda-1)]} \equiv \bar{M}^L(\lambda)$

Proof. Following Equation (8) and due to the fact that $V(0) \geq 0$, I can bound $D^L(\lambda)$ from below: $D^L(\lambda) \geq \frac{1-\delta\lambda-\delta p(1-\lambda)}{(1-\delta\lambda)(1-\delta p)+\delta p(1-\delta)(1-\lambda)}x$. Now, $M^L(\lambda) = \frac{x-(1-p)D^L(\lambda)}{(1+p)x-(1-p)D^L(\lambda)} \leq \frac{(1-\delta\lambda)(1-\delta)+\delta(1-\lambda)(2-\delta-p)}{(2-\delta-\delta p)[1-\delta(2\lambda-1)]}$. □

I now turn to the proof of Proposition 5:

Proposition 5. (Free-Riding and Gradual Entry)

Assume both agents share the same belief and Loner is out of the market. If λ is close enough to 1 then Follower will enter the market strictly after Loner.

Proof.

Loner will start investing in exactly n periods if and only if his belief today is in the region $\Big[0.5 - \frac{0.5-M^L}{(2\lambda-1)^n}, 0.5 - \frac{0.5-M^L}{(2\lambda-1)^{n-1}}\Big)$. Following Lemma 15, in order to show that Follower will not enter before that, it suffices to show that $z^{n-1} \equiv 0.5 - \frac{0.5-M^L}{(2\lambda-1)^{n-1}} \leq y_n$ for all $n = 0, ..., \bar{n}+1$. The proof will be conducted by induction.

For n=0:

For the purpose of this proof denote thresholds as functions of λ, i.e., $M^L(\lambda)$ and $M_0^F(\lambda)$. I need to show that $(2\lambda-1)M^L(\lambda) + (1-\lambda) \leq M^F(\lambda)$ for λ close enough to 1. The inequality is strong for $\lambda = 1$ since $M^L(1) = \frac{1-\delta}{2-\delta(1+p)} < \frac{1-\delta p}{2-\delta p(1+p)} = M_0^F(1)$ (Propositions 1 and 2). Thus, from the continuity of thresholds at $\lambda = 1$ (Lemma 14), it follows that the inequality holds in an environment of $\lambda = 1$.

For $n+1 \geq 1$:

Note that $y^{n+1} = \frac{x-(1-p)D^{n+1}}{(1+p)x-(1-p)D^{n+1}} \geq z^n$ if and only if $D^{n+1} \leq \frac{1-(1+p)z^n}{(1-p)(1-z^n)}x$.

$$D^{n+1} = x - p\delta(1-\lambda)\alpha^n + p\delta\big[U(1-\lambda,n) - \beta^n\big] =$$

$$x + p\delta\big[U(0,n) - \beta^n\big] - p\delta(1-\lambda)\left[\alpha^n - \frac{U(1-\lambda,n) - U(0,n)}{1-\lambda}\right] \underbrace{\leq}_{U \text{ is convex}}$$

$$x + p\delta\big[U(0,n) - \beta^n\big] - p\delta(1-\lambda)\left[\alpha^n - \frac{\alpha^n 0.5 + \beta^n - U(0,n)}{0.5}\right] =$$

$$x + p\delta(2\lambda-1)\big[U(0,n) - \beta^n\big] = x + p\delta(2\lambda-1)D^n \underbrace{\leq}_{\text{ind. hyp.}}$$

$$x\left[1 + p\delta(2\lambda-1)\frac{1-(1+p)z^{n-1}}{(1-p)(1-z^{n-1})}\right] \underbrace{=}_{z^{n-1}=(2\lambda-1)z^n+1-\lambda} =$$

$$x\left[1 + p\delta(2\lambda-1)\frac{1-(1+p)\big[(2\lambda-1)z^n + 1 - \lambda\big]}{(1-p)\big[\lambda - (2\lambda-1)z^n\big]}\right]$$

It thus remains to show that:

$$1 + p\delta(2\lambda-1)\frac{1-(1+p)\big[(2\lambda-1)z^n + 1 - \lambda\big]}{(1-p)\big[\lambda - (2\lambda-1)z^n\big]} \leq \frac{1-(1+p)z^n}{(1-p)(1-z^n)}$$

Rearranging this inequality I get that I need to prove:

$$\forall n \in \{1,...,\bar{n}+1\} \quad g(z^n) \equiv 1 - \delta(2\lambda-1) + p\delta(2\lambda-1)\frac{z^n + \frac{1-\lambda}{2\lambda-1}}{1 - z^n + \frac{1-\lambda}{2\lambda-1}} - \frac{z^n}{1-z^n} \geq 0 \qquad (10)$$

I will prove that $g(z) \geq 0$ for all $z \in [0, M^L]$. Note that $g'(z) = \frac{p\delta}{\left[\frac{\lambda}{2\lambda-1} - z\right]^2} - \frac{1}{[1-z]^2} \leq 0$ since $p\delta \leq 1$ and $\frac{\lambda}{2\lambda-1} \geq 1 \geq z$. Thus, in order to prove (10) it suffices to show that $g(\bar{M^L}) \geq 0$ where $\bar{M^L}$ is the bound defined in Lemma 16.

$g(z) \geq 0$ if and only if

$$\Big[1-\delta(2\lambda-1)\Big]\left[1 - z + \frac{1-\lambda}{2\lambda-1}\right]\Big[1-z\Big] + p\delta(2\lambda-1)\left[z + \frac{1-\lambda}{2\lambda-1}\right]\Big[1-z\Big] - z\left[1 - z + \frac{1-\lambda}{2\lambda-1}\right] \geq 0$$

Rearranging this inequality yields:

$$\underbrace{\Big[1 - \delta - \Big(2 - \delta(1+p)\Big)z\Big]}_{A(z)} + \underbrace{(1-\lambda)\frac{1-2z}{1-z}\left[\frac{1}{2\lambda-1} + \delta(1+p)(1-z)\right]}_{B(z)} \geq 0 \qquad (11)$$

Now,

$$A(\bar{M^L}) = 1 - \delta - \frac{(1-\delta\lambda)(1-\delta) + \delta(1-\lambda)(2-\delta-p)}{1-\delta(2\lambda-1)} =$$

$$\frac{(1-\delta\lambda)(1-\delta) - \delta(1-\lambda)(2-\delta-p)}{1-\delta(2\lambda-1)}$$

$$B(\bar{M^L}) = (1-\lambda)\frac{\delta(1-\delta)(1-p)(2\lambda-1)}{(1-\delta\lambda)(1-\delta p)+\delta p(1-\delta)(1-\lambda)}\left[\frac{1}{2\lambda-1}+\delta(1+p)(1-\bar{M^L})\right] \geq$$
$$(1-\lambda)\frac{\delta(1-\delta)(1-p)(2\lambda-1)}{1-\delta\lambda+(1-\delta)(1-\lambda)}\left[\frac{1}{2\lambda-1}+0\right] = \frac{\delta(1-\delta)(1-p)(1-\lambda)}{1-\delta(2\lambda-1)}$$

Thus,

$$A(\bar{M^L})+B(\bar{M^L}) \geq \frac{(1-\delta\lambda)(1-\delta)-\delta(1-\lambda)(2-\delta-p)+\delta(1-\delta)(1-p)(1-\lambda)}{1-\delta(2\lambda-1)} =$$
$$\frac{(1-\delta)^2-\delta^2(1-p)(1-\lambda)}{1-\delta(2\lambda-1)}$$

It follows that inequality (11) is satisfied for $\bar{M^L}$ if $(1-p)(1-\lambda) \leq \left(\frac{1-\delta}{\delta}\right)^2$.

□

A.4 Proofs on Private vs. Public Information

A.4.1 Proof of Proposition 8

Similarly to the proof of Proposition 4.2, in order to show that $\tilde{U}(\mu) = \big(U(\mu,0),...,U(\mu,\bar{n})\big)'$ is smaller than $\tilde{U}_B(\mu) \equiv \big(U_B(\mu,0),...,U_B(\mu,\bar{n})\big)'$, it suffices to show that if $\tilde{f}(\mu) = (f(\mu,0),...,f(\mu,\bar{n}))' \leq \tilde{U}_B(\mu)$ then $\tilde{T}_F(\tilde{f})(\mu) \leq \tilde{U}_B(\mu)$ as well (see proof of Proposition 4.2 for more details).

Since T_L is monotone then $\tilde{T}_F(\tilde{f}(\mu))_n = T_L(f(\mu,n-1)) \leq T_L(U_B(\mu,n-1)) = U_B(\mu,n)$ for all $n = 1,...,\bar{n}$. As for the first argument of $\tilde{T}_F(\tilde{f})(\mu)$, it is the maximum of two arguments:

$$\delta\sum_{s\in S} f(s)Q^N\Big(s,(\mu,0)\Big) =$$
$$[p+(1-p)\mu]\delta f\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda,0\right)+(1-p)(1-\mu)\delta f(1-\lambda,\bar{n}) \leq$$
$$[p+(1-p)\mu]\delta U_B\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu}+1-\lambda,0\right)+(1-p)(1-\mu)\delta U_B(1-\lambda,\bar{n}) \underbrace{\leq}_{U_B\ convex}$$
$$p\delta U_B\Big((2\lambda-1)\mu+1-\lambda,0\Big)+(1-p)\delta\Big[\mu U_B(\lambda,0)+(1-\mu)U_B(1-\lambda,\bar{n})\Big] = U_B^N(\mu,0)$$

and,

$$(2\mu-1)x+\delta\sum_{s\in S} f(s)Q^I\Big(s,(\mu,0)\Big) =$$
$$(2\mu-1)x+p\left[\delta\sum_{s\in S} f(s)Q^N\Big(s,(\mu,0)\Big)\right]+(1-p)\mu\delta f(\lambda,0)+$$
$$(1-p)(1-\mu)\delta\Big[pf(1-\lambda,0)+(1-p)f(1-\lambda,\bar{n})\Big] \leq$$
$$(2\mu-1)x+pU_B^N(\mu,0)+(1-p)\mu\delta U_B(\lambda,0)+$$
$$(1-p)(1-\mu)\delta\Big[pU_B(1-\lambda,0)+(1-p)U_B(1-\lambda,\bar{n})\Big] = U_B^I(\mu,0)$$

Thus, the first argument of $\tilde{T}_F(\tilde{f})(\mu)$ is smaller than $\max\{U_B^N(\mu,0),U_B^I(\mu,0)\} = U_B(\mu,0)$.

A.4.2 Proof of Proposition 9

The proof requires three steps:

1. $M_0^{FB}(\lambda)$ is continuous at $\lambda = 1$
2. $y^{Bn} \equiv \frac{x-(1-p)[U^B(0,n)-\beta^{Bn}]}{(1+p)x-(1-p)[U^B(0,n)-\beta^{Bn}]} \leq M_n^{FB}$
3. $y^{Bn} \geq z^{n-1} \equiv 0.5 - \frac{0.5-M^L}{(2\lambda-1)^{n-1}}$ for λ close enough to 1.

The proofs are similar to the ones in the private information case (Appendix A.3) so I leave out some details.

1. Define
$$W_B(\mu) = \max\{W_B^N(\mu), W_B^I(\mu)\}$$
where,
$$W_B^N(\mu) = p\delta W_B\Big((2\lambda-1)\mu + 1 - \lambda\Big) + (1-p)\delta\Big[\mu W_B(\lambda) + (1-\mu)W_B(1-\lambda)\Big]$$
$$W_B^I(\mu) = (2\mu-1)x + pW_B^N(\mu) + (1-p)\delta\Big[\mu W_B(\lambda) + (1-\mu)W_B(1-\lambda)\Big]$$
Similarly to the proof of Proposition 4 (Appendix A.2) it can be shown that W_B has the same threshold as U_B and it is equal to $M_0^{FB} = \frac{x-(1-p)[W_B(0)-\beta^{FB}]}{(1+p)x-(1-p)[W_B(0)-\beta^{FB}]}$ where $\beta^{FB} = \frac{x-(1-\delta\lambda)\alpha^{FB}}{1-\delta}$ and $\alpha^{FB} = \frac{x(2-\delta-\delta p^2)-(1-\delta)(1-p^2)W_B(0)}{(1-\delta\lambda)(1-\delta p^2)+\delta p^2(1-\delta)(1-\lambda)}$.
Similarly to the arguments given in Appendix A.3 it can be shown that if we think of $W_B(0)$ as a function of λ then it is continuous at $\lambda = 1$, and so is M_0^{FB}.
2. Follows from the proof of Lemma 15 by replacing U with U^B.
3. The proof is conducted by induction. For $n = 0$, due to the continuity of $M_0^F(\lambda)$ and $M^L(\lambda)$ at $\lambda = 1$, it suffices to show that $M^L(1) = \frac{1-\delta}{2-\delta(1+p)} < M_0^{FB}(1)$. Note that if $\lambda = 1$ then $W_B(0) = 0$. By inserting this in the solution of M_0^{FB} from item 1 we get that $M_0^{FB}(1) = \frac{1-\delta p}{2-\delta p(1+p)}$ which is in fact larger than $M^L(1)$. The proof for $n \geq 1$ follows from the proof of Proposition 5 (Appendix A.3) by replacing U, α^n and β^n with U^B, α^{Bn} and β^{Bn}.

A.4.3 Proof of Proposition 10

1. The fact that $M_0^{FB} \leq M_0^F$ will follow from two claims: (1) $W_B \geq W$ and (2) $W_B(0) - \beta^B \geq W(0) - \beta^F$.

   (1) Denote by T_B the mapping whose fixed point is W_B. It suffices to show that if $f \geq W$ then $T_B(f) \geq W$ as well (see footnote 7). $T_B(f)$ is the maximum of two elements:
$$f^N(\mu) \equiv p\delta f\Big((2\lambda-1)\mu + 1 - \lambda\Big) + (1-p)\delta\Big[\mu f(\lambda) + (1-\mu)f(1-\lambda)\Big] \geq$$
$$p\delta W\Big((2\lambda-1)\mu + 1 - \lambda\Big) + (1-p)\delta\Big[\mu W(\lambda) + (1-\mu)W(1-\lambda)\Big] \underbrace{\geq}_{W convex}$$
$$[p+(1-p)\mu]\delta W\left((2\lambda-1)\frac{\mu}{p+(1-p)\mu} + 1 - \lambda\right) + (1-p)(1-\mu)\delta W(1-\lambda) = W^N(\mu)$$

$$f^I(\mu) \equiv (2\mu - 1)x + pf^N(\mu) + \delta(1-p)\Big[\mu f(\lambda) + (1-\mu)f(1-\lambda)\Big] \geq$$
$$(2\mu - 1)x + pW^N(\mu) + \delta(1-p)\Big[\mu W(\lambda) + (1-\mu)W(1-\lambda)\Big] = W^I(\mu)$$

It follows that $T_B(f)(\mu) = \max\{f^N(\mu), f^I(\mu)\} \geq \max\{W^N(\mu), W^I(\mu)\} = W(\mu)$

(2) Using (1):

$$\begin{aligned} W_B(0) - \beta^B &= W_B(0) - \frac{x - (1-\delta\lambda)\alpha^B}{1-\delta} = \\ & W_B(0)\left[1 - \frac{(1-\delta\lambda)(1-\delta)(1-p^2)}{(1-\delta\lambda)(1-\delta p^2) + \delta p^2(1-\delta)(1-\lambda)}\right] + \\ & \frac{1 - \delta\lambda - \delta p^2(1-\lambda)}{(1-\delta\lambda)(1-\delta p^2) + \delta p^2(1-\delta)(1-\lambda)}x \geq \\ & W(0)\left[1 - \frac{(1-\delta\lambda)(1-\delta)(1-p^2)}{(1-\delta\lambda)(1-\delta p^2) + \delta p^2(1-\delta)(1-\lambda)}\right] + \\ & \frac{1 - \delta\lambda - \delta p^2(1-\lambda)}{(1-\delta\lambda)(1-\delta p^2) + \delta p^2(1-\delta)(1-\lambda)}x = W(0) - \beta^F \end{aligned}$$

2. If $\lambda = 1$ then claim (2) holds with equality since $W(0) = W_B(0) = 0$. In this case $M_0^{FB} = M_0^F = \frac{1-\delta p}{2-\delta p(1+p)}$.

3. Assume that Loner entered in some period that we normalize to 0. In equilibrium all agents share the same belief at this point, call it μ. Consider n periods after Loner's entry, Loner is in the market and both types of followers were out at $n-1$. Since Follower A observed Loner stay in the market for n periods, his belief in period n is $\chi_n \equiv \frac{\mu}{p^n + (1-p^n)\mu}$. As for Follower B, if Loner received a good signal π^G in period n then Follower B updates his belief to λ and enters, i.e., he enters (weakly) before Follower A. Otherwise, so far Loner received only uninformative signals of zero and Follower B has a belief of $\zeta_n(\lambda) \equiv 0.5 - (2\lambda - 1)^n(0.5 - \mu)$. I will show that

$$M_0^F(\lambda) - \chi_n \leq M_0^{FB}(\lambda) - \zeta_n(\lambda) \tag{12}$$

This implies that if Follower A doesn't enter in n (i.e., $M_0^F > \chi_n$), then Follower B also doesn't enter (i.e., $M_0^{FB} > \zeta_n(\lambda)$). Recall that $M_0^F = M_0^{FB}$ when $\lambda = 1$. Furthermore, $\zeta_n(1) = \mu < \frac{\mu}{p^n + (1-p^n)\mu} = \chi_n$. Thus, inequality (12) is strong when $\lambda = 1$ and from the continuity of all elements (similarly to Lemma 14 it can be shown that M_0^{FB} is continuous at $\lambda = 1$), it holds in an environment of $\lambda = 1$.

□

A.4.4 Proof of Corollary 11

Following Proposition 10.4 I need to show that the probability that Loner observes π^G before Follower A enters is smaller than 0.5. Assume Follower A enters k periods after Loner (in the leading example in the body of the paper $k = 2$). Let p_k^θ denote the probability Loner observed π^G in those k periods, without observing π^B before that, given that the state was θ when he entered:

$$p_k^\theta = \Pr\left(\pi_t^L = ... = \pi_{t+i-1}^L = 0\,,\, \pi_{t+i}^L = \pi^G for\, some\, i \leq k \middle| \theta_t = \theta,\, Loner\ entered\ in\ t\right)$$

I need to show that $\sum_{\theta=G,B} p_k^\theta \Pr\left(\theta_t = \theta | \textit{Loner entered in } t\right) < 0.5$. Note that p_k^G and p_k^B have the following recursive structure:

$$\begin{aligned}
p_k^G =& 1 - p + p\left[\lambda p_{k-1}^G + (1-\lambda) p_{k-1}^B\right] \\
p_k^B =& p\left[\lambda p_{k-1}^B + (1-\lambda) p_{k-1}^G\right] \\
p_0^G =& p_0^B = 0
\end{aligned}$$

The solution for this system is:

$$\begin{aligned}
p_k^G =& \frac{2 - p^k(1+p) - (2\lambda-1)^k p^k (1-p) - 2\lambda p(1-p^k)}{2\left[1-(2\lambda-1)p\right]} \\
p_k^B =& \frac{2p - p^k(1+p) + (2\lambda-1)^k p^k (1-p) - 2\lambda p(1-p^k)}{2\left[1-(2\lambda-1)p\right]}
\end{aligned}$$

These expressions increase in k so $p_k^G \leq \frac{1-p+(1-\lambda)p}{1-(2\lambda-1)p}$, $p_k^B \leq \frac{(1-\lambda)p}{1-(2\lambda-1)p}$. Since in equilibrium Loner enters when the probability for state G is smaller than 0.5,

$$\sum_{\theta=G,B} p_k^\theta \Pr\left(\theta_t = \theta | \textit{Loner entered in } t\right) < 0.5 \frac{1-p+(1-\lambda)p}{1-(2\lambda-1)p} + 0.5 \frac{(1-\lambda)p}{1-(2\lambda-1)p} = 0.5$$

□